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Optimal interval observers for discrete-time linear switched systems

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ABSTRACT

The main goal of this paper is to present a methodology to design interval observers for discrete-time linear switched systems affected by bounded, but unknown disturbances. Two design techniques are presented. The first one requires that the observation error dynamics are nonnegative while the second one relaxes this restrictive requirement by a change of coordinates. Furthermore, ideas of using H_∞ formalism to compute optimal gains are proposed. Finally, illustrative examples highlight the performance of our methodology.

KEYWORDS

Interval observer; discrete-time; linear switched systems; LMI; H_∞ design

1. Introduction

One solution to estimate the state variables of a system when some of them are not available for measurements is to use a real-time estimation algorithm. For monitoring, fault detection and feedback control purposes, all of state variables must be available and observers have demonstrated the ability to reproduce efficiently this expected information. Therefore state estimation has become a fundamental problem in control theory and has been developed in many directions. A traditional estimator is the Luenberger observer (Luenberger, 1971) which computes point estimates of the state from input-output data. However, in some cases, this technique may not provide component-wise information of the state vector due to uncertainties. In the last two decades, a new technique of state estimation has been proposed to meet the practical demand. It is based on the notions of framers and interval observers (Gouzé et al., 2000). Framers and interval observers belong to a specific class of estimators called guaranteed state estimation methods whose strength is to provide a region of the state space where the unknown variables are sure to belong. Framers and interval observers are composed of a dynamic extension with two outputs giving upper and lower bounds for the solutions of the considered system at each instant. More precisely, an upper and a lower bound are provided for each component of the state and, in the absence of disturbances, the

norm of the error between the bounds converges to zero. There are two other reasons why framers and interval observers become more and more popular. First, they make it possible to cope with large uncertainties, which is very important for example when we consider biological models. Second, they have been successfully applied to many real-life problems (see e.g., (Bernard & Gouzé, 2004), (Alcaraz-Gonzalez et al, 2007), (Goffaux et al., 2009) and references therein). Framers and interval observers have been proposed in many contributions for both continuous-time and discrete-time systems. Some works are devoted to various classes of finite or infinite-dimensional linear systems (Mazenc & Bernard, 2010), (Mazenc & Bernard, 2011), (Mazenc et al., 2014), (Mazenc & Dinh, 2014), (Efimov et al., 2013), (Loukkas et al., 2017), bilinear systems (Dinh & Ito, 2016) and others concern some classes of nonlinear systems (Moisan et al., 2009), (Raïssi et al., 2012), (Raïssi et al., 2005), (Mazenc et al., 2013), (Ito & Dinh, 2018).

Recently, in the context of systems that exhibit changes along the time among a finite number of possible dynamical behaviors (i.e. switched systems), framers and interval observers have started to be designed (He & Xie, 2016), (Ethabet et al., 2017), (Haifa et al., 2018), (Briat & Khammash, 2017), (Ifqir et al., 2017), (Guo & Zhu, 2017). The study of switched systems has received growing attention which can be explained by the fact of the different application domains treated as switched systems, such as the control of mechanical systems, the automotive industry and the automatic process control (Branicky, 1998), (Lin et al., 2009). Actually, most of interval observer designs in this context are for continuous-time switched systems. To the best of the authors' knowledge, the case of discrete-time switched systems, as e.g., in (Guo & Zhu, 2017), has not been fully considered in the literature. In the present paper, the main objective is to design interval observers for the family of discrete-time linear switched systems affected by bounded but unknown disturbances. Two design techniques are mentioned. The first one requires that the observation error dynamics are nonnegative while the second one relaxes this restrictive requirement by a change of coordinates. It is worth pointing out that the designs we propose are not derived directly from the interval observers constructed for continuous-time systems in (Dinh et al., 2014), (Ethabet et al., 2017), (Haifa et al., 2018), although some of the key ideas of these works are used along our construction. In fact, changing the system from non-switched case to switched case and from continuous time to discrete time raises the radical changes of stability properties. Consequently, new criteria of stability need to be stated and proved. Additionally, our designs differ from (Guo & Zhu, 2017): the interval observer we propose is simpler in its dynamics. Each copy of observer, or its associated error equation, does not possess the property of being a cooperative or a nonnegative system. This fact is the crucial difference compared with the designs of interval observers introduced in (Guo & Zhu, 2017) which are carried out for cooperative systems after a coordinate transformation. In fact, we will use the notion of nonnegative and cooperative system as well, but only indirectly to select for the interval observer appropriate initial conditions and upper and lower bounds for the solutions of the studied system. The simplicity of introduced interval observer designs in the present paper not only makes the stability analysis easier, but also allows one to avoid the hybrid behaviour due to changes of coordinates. In addition to their simplicity, the interval observers we present offer the possibility to construct a bundle of interval observers, as done for instance in (Bernard & Gouzé, 2004), without having to introduce extra dynamics, simply by proposing several choices of initial conditions and bounding outputs. Furthermore, this paper extends the preliminary work introduced in (Marouani et al., 2018). The most important improvement with respect to (Marouani

et al., 2018) is the idea of using H_∞ formalism to compute optimal gains. In fact until now, the problem of optimizing the accuracy of the interval between upper and lower bounds is not yet fully investigated, e.g., in (Wang et al., 2018), H_∞ technique has been considered in interval observer design for discrete-time linear systems. We bring in this paper a solution by employing bounded-real lemma to design H_∞ interval observer to obtain a tighter interval width for a class of complex systems. Comparative simulations are given to illustrate these ideas.

The paper is organized as follows. The preliminaries with the introduction of a definition of interval observers are given in Section 2. The main results are stated and proved in Section 3. Comparative simulations are given in Section 4. Finally, a conclusion is drawn in Section 5.

2. Preliminaries

2.1. Notation, definitions, basic result

The set of natural numbers, integers and real numbers are denoted by \mathbb{N} , \mathbb{Z} and \mathbb{R} , respectively. The set of nonnegative real numbers and nonnegative integers are denoted by $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$, respectively. For a vector $x \in \mathbb{R}^n$, the Euclidean norm is denoted by $|x|$. For a signal $x(k) : N \rightarrow \mathbb{R}^n$, the \mathcal{L}_2 norm is denoted by $\|x(k)\|_2$. For a measurable and locally essentially bounded input $u : \mathbb{Z} \rightarrow \mathbb{R}$, the symbol $\|u\|_{[t_0, t_1]}$ denotes its \mathcal{L}_∞ norm:

$$\|u\|_{[t_0, t_1]} = \sup\{|u|, t \in [t_0, t_1]\}.$$

If $t_1 = \infty$ then we will simply write $\|u\|$. We denote \mathcal{L}_∞ as the set of all inputs u with the property $\|u\| < \infty$. We denote the sequence of integers $1, \dots, N$ as $\overline{1, N}$. Inequalities must be understood *component-wise*, i.e., for $x_a = [x_{a,1}, \dots, x_{a,n}]^\top \in \mathbb{R}^n$ and $x_b = [x_{b,1}, \dots, x_{b,n}]^\top \in \mathbb{R}^n$, $x_a \leq x_b$ if and only if, for all $i \in \overline{1, n}$, $x_{a,i} \leq x_{b,i}$. For a matrix $Q \in \mathbb{R}^{m \times n}$, define Q^+ , $Q^- \in \mathbb{R}^{m \times n}$ such as $Q^+ = \max(Q, 0)$ and $Q^- = Q^+ - Q$ and the matrix of absolute values of all elements be defined by $|Q| = Q^+ + Q^-$, the superscripts $+$ and $-$ for other purposes are defined appropriately when they appear. A square matrix $Q \in \mathbb{R}^{n \times n}$ is said to be nonnegative if all its entries are nonnegative. I_n is the identity matrix of $n \times n$ dimension. Any $n \times m$ (resp. $p \times 1$) matrix, whose entries are all 1 is denoted $E_{n \times m}$ (resp. E_p) and whose entries are all 0 is denoted $0_{n \times m}$ (resp. 0_p). The vector of eigenvalues of each matrix $A \in \mathbb{R}^{n \times n}$ is denoted by $\lambda(A)$. $P \in \mathbb{R}^{n \times n}$ is positive (resp. negative) (semi-)definite is denoted as $P \succ (\succcurlyeq) 0$ (resp. $P \prec (\preccurlyeq) 0$).

Lemma 1. (Efimov et al., 2012) Consider a vector $x \in \mathbb{R}^n$ such that $\underline{x} \leq x \leq \bar{x}$ for some $\underline{x}, \bar{x} \in \mathbb{R}^n$, and a constant matrix $A \in \mathbb{R}^{m \times n}$, then

$$A^+ \underline{x} - A^- \bar{x} \leq Ax \leq A^+ \bar{x} - A^- \underline{x}. \quad (1)$$

Lemma 2 (Schur Complement (Boyd et al, 1994)). Given the matrices $A = A^T$, $C = C^T$ and B with appropriate dimensions. The following LMIs are equivalent:

(1)

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0.$$

(2)

$$C \succ 0; A - BC^{-1}B^T \succ 0.$$

(3)

$$A \succ 0; C - B^T A^{-1}B \succ 0.$$

A discrete-time system described by $x(k+1) = f(x(k))$ is nonnegative if for any integer k_0 and any initial condition $x(k_0) \geq 0$, the solution x satisfies $x(k) \geq 0$ for all integers $k \geq k_0$.

As a consequence of the definition of nonnegative systems right above, a linear system described by $x(k+1) = Ax(k) + u(k)$, with $x(k) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$, is nonnegative if and only if the matrix A is elementwise nonnegative, $u(k) \geq 0$ and $x(k_0) \geq 0$. In this case, the system is also called cooperative. This property is essential in the design of interval observers since the estimation errors should follow nonnegative dynamics.

2.2. Interval observer for switched systems

Definition 1. Consider a switched system:

$$\begin{cases} x(k+1) = f_q(x(k), d(k)), \\ y(k) = g_q(x(k)), \end{cases} \quad (2)$$

with the state $x \in \mathbb{R}^n$, the output $y \in \mathbb{R}^p$, the index of the active subsystem $q \in \overline{1, N}$, the number of subsystems is $N \in \mathbb{N}$ and f_q, g_q are functions. The uncertainties $d(k) \in \mathbb{R}^\ell$ are such that there exists a sequence $\bar{d}(k) \in \mathbb{R}^\ell$ such that, for all $k \geq 0$, $-\bar{d}(k) \leq d(k) \leq \bar{d}(k)$. The initial condition $x(0)$, is assumed to be bounded by two known bounds:

$$\underline{x}(0) \leq x(0) \leq \bar{x}(0). \quad (3)$$

Then, the dynamical system

$$z(k+1) = h_q(z(k), y(k), \bar{d}(k)), q \in \overline{1, N}, N \in \mathbb{N}, \quad (4)$$

associated with the initial condition $z(0) = r_q(\bar{x}(0), \underline{x}(0)) \in \mathbb{R}^{n_z}$ and bounds for the solution $x(k)$: $\bar{x}(k) = \bar{h}_q(z(k), y(k))$, $\underline{x}(k) = \underline{h}_q(z(k), y(k))$, where $q \in \overline{1, N}$, $N \in \mathbb{N}$, h_q, r_q, \bar{h}_q and \underline{h}_q are functions, is called

(i) a framer for (2) if for any vectors $x(0), \underline{x}(0)$ and $\bar{x}(0)$ in \mathbb{R}^n satisfying (3), the solutions denoted respectively x and z of (2)-(4) with respectively $x(0), z(0) = r_q(\bar{x}(0), \underline{x}(0))$ as initial condition at 0, satisfy for all $k \geq 0$, the inequalities

$$\underline{x}(k) = \underline{h}_q(z(k), y(k)) \leq x(k) \leq \bar{h}_q(z(k), y(k)) = \bar{x}(k), \quad (5)$$

(ii) an *interval observer* for (2) if in addition $|\bar{h}_q(z(k), y(k)) - \underline{h}_q(z(k), y(k))|$ is input-to-state stable (ISS) with respect to $d(k) \in \mathbb{R}^\ell$ for all $q \in \overline{1, N}$, $N \in \mathbb{N}$.

3. Main results

Consider the following discrete-time linear switched system:

$$\begin{cases} x(k+1) = A_q x(k) + B_q u(k) + d(k) \\ y(k) = C_q x(k) \end{cases}, \quad q \in \overline{1, N}, \quad N \in \mathbb{N}, \quad (6)$$

with $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, $d \in \mathbb{R}^\ell$ is the disturbances. q is the index of the active subsystem and N is the number of subsystems. A_q , B_q and C_q are time-invariant matrices of the corresponding dimensions.

In this section, the goal is to design framers and interval observers for discrete-time linear switched systems. First, two approaches to construct framers are considered: the first one is based on the nonnegativity (cooperativity) of the estimation errors in the original coordinates, while the second is more general and follows a transformation of coordinates. Using the available information and considering that $\underline{x}(0) \leq x(0) \leq \bar{x}(0)$ for some known $\underline{x}(0), \bar{x}(0) \in \mathbb{R}^n$, the objective is to calculate two estimates $\underline{x}, \bar{x} \in \mathcal{L}_\infty^n$, such that

$$\underline{x}(k) \leq x(k) \leq \bar{x}(k), \quad k \in \mathbb{Z}_+. \quad (7)$$

Later, the gains can be computed by H_∞ formalism which turn framers into interval observers and allow to cope with uncertainties. In addition, thank to this design, we can improve accuracy of the interval observers because the width of enclosure depends on selecting the observer gains.

Some assumptions are introduced for the rest of the paper.

Assumption 1. *The state vector $x \in \mathbb{R}^n$ is bounded, i.e. $x \in \mathcal{L}_\infty^n$.*

Assumption 2. *Let a function $\bar{d} \in \mathcal{L}_\infty^n$ such that for all $k \in \mathbb{Z}_+$*

$$-\bar{d}(k) \leq d(k) \leq \bar{d}(k).$$

Assumption 1 states that the state x is bounded. This assumption is common in the theory of observers design since the control design (stabilization) is not considered at this stage. Assumption 2 is basic in the literature of interval observers where the disturbances are assumed bounded with known bounds.

3.1. Framer design in the original coordinates

In this part, we introduce the following assumption in order to design the framer without a transformation of coordinates:

Assumption 3. *There exist $L_q \in \mathbb{R}^{n \times p}$ such that $A_q - L_q C_q$ are nonnegative, for all $q \in \overline{1, N}$.*

Assumption 3 is an important condition to design an interval observer, it is rather restrictive and it will be relaxed in Section 3.2.

As a solution to (7), the following framer candidate is considered:

$$\begin{cases} \bar{x}(k+1) = (A_q - L_q C_q)\bar{x}(k) + B_q u(k) + \bar{d}(k) + L_q y, \\ \underline{x}(k+1) = (A_q - L_q C_q)\underline{x}(k) + B_q u(k) - \bar{d}(k) + L_q y, \\ \underline{x}(0) = \underline{X}_0, \bar{x}(0) = \bar{X}_0, \end{cases} \quad (8)$$

where $L_q \in \mathbb{R}^{n \times p}$ is an appropriate observer gain associated to the q -subsystem with $q \in \overline{1, N}$, which satisfies Assumption 3 and later in Section 3.3 ensures that (8) becomes an interval observer.

Theorem 1 (First framer design). *Let Assumptions 1-3 hold, the lower and upper bounds $\underline{x}(k)$, $\bar{x}(k)$ for the state $x(k)$ given by (8) satisfy (7), provided that $\underline{X}_0 \leq x(0) \leq \bar{X}_0$.*

Proof. Let $\bar{e}(k) = \bar{x}(k) - x(k)$ and $\underline{e}(k) = x(k) - \underline{x}(k)$ be the upper observation and the lower observation errors, respectively. The aim is to prove that $\bar{e}(k)$ and $\underline{e}(k)$ are nonnegative and bounded. The dynamics of the upper error follow:

$$\bar{e}(k+1) = (A_q - L_q C_q)\bar{e}(k) + \bar{d}(k) - d(k), \quad (9)$$

Similarly, the dynamics of the lower error are described by:

$$\underline{e}(k+1) = (A_q - L_q C_q)\underline{e}(k) + \bar{d}(k) + d(k), \quad (10)$$

According to Assumption 2, we have $\bar{d}(k) + d(k) \geq 0$ and $\bar{d}(k) - d(k) \geq 0$. Bearing in mind Assumption 3 and from the fact that $\bar{e}(0) = \bar{x}(0) - x(0) \geq 0$ and $\underline{e}(0) = x(0) - \underline{x}(0) \geq 0$, it follows that, for all $k \in \mathbb{Z}_+$, $\bar{e}(k) \geq 0$ and $\underline{e}(k) \geq 0$. Thus, for all $k \in \mathbb{Z}_+$, $\underline{x}(k) \leq x(k) \leq \bar{x}(k)$. This allows us to conclude that (8) is a framer for (6). \square

3.2. Framer design in the new coordinates

Even though we have proposed in Theorem 1 the different steps to design a framer, in some cases, it is not possible to find gains L_q such that the matrices $A_q - L_q C_q$ are nonnegative, i.e. Assumption 3 is restrictive. Naturally, one can think about finding a nonsingular transformation $z = Rx$ such that the matrices $R(A_q - L_q C_q)R^{-1}$ are nonnegative. Subsequently, framer can be constructed in these new coordinates. However, the existence of a common transformation R for all $q \in \overline{1, N}$ is not obvious, even impossible.

A new methodology is proposed. It is based on the design, in the original base, of two conventional observers. The structure is inspired by the one proposed in (Dinh et al., 2014) for non-switched systems. We introduce the assumption needed in the following.

Assumption 4. *There exist changes of coordinates R_q , $q = \overline{1, N}$ such that the matrices*

$$R_q(A_q - L_q C_q)R_q^{-1} \quad (11)$$

are nonnegative for all $q = \overline{1, N}$.

Theorem 2 (Second framer design). *Let Assumptions 1-2 and 4 hold. Consider the discrete-time linear switched system (6) and the dynamic extension as follows:*

$$\begin{cases} \hat{x}^+(k+1) &= (A_q - L_q C_q) \hat{x}^+(k) + B_q u(k) + R_q^{-1} |R_q| \bar{d}(k) + L_q y(k), \\ \hat{x}^-(k+1) &= (A_q - L_q C_q) \hat{x}^-(k) + B_q u(k) - R_q^{-1} |R_q| \bar{d}(k) + L_q y(k), \end{cases} \quad (12)$$

associated with the suitably selected initial conditions:

$$\begin{cases} \hat{x}^+(0) = S_q(R_q^+ \bar{x}(0) - R_q^- \underline{x}(0)), \\ \hat{x}^-(0) = S_q(R_q^+ \underline{x}(0) - R_q^- \bar{x}(0)), \end{cases} \quad (13)$$

where

$$S_q = R_q^{-1}, \quad \forall q \in \overline{1, N}, \quad N \in \mathbb{N}. \quad (14)$$

Then,

$$\begin{cases} \underline{x}(k) = S_q^+ R_q \hat{x}^-(k) - S_q^- R_q \hat{x}^+(k), \\ \bar{x}(k) = S_q^+ R_q \hat{x}^+(k) - S_q^- R_q \hat{x}^-(k), \end{cases} \quad (15)$$

is a framer for (6) satisfying (7).

Remark 1. *Assumptions 3 and 4, which are related to the notion of nonnegative and cooperative system, are fundamental and frequently used in designing interval observers. Assumption 4 allows one to relax the restrictive Assumption 3. In fact thank to changes of coordinates, L_q in (12) will be selected only for purpose of stability instead of having to ensure both stability and nonnegativity constraints as in (8). The Section 3.3 introduces the main theorem as guidelines for selecting L_q for (8) and (12).*

Proof. Let us prove that $\bar{x}(k) - x(k) \geq 0$ and $x(k) - \underline{x}(k) \geq 0$. First, let us define errors $E_q^+(k)$ and $E_q^-(k)$ as:

$$E_q^+(k) = R_q \hat{x}^+(k) - R_q x(k), \quad (16)$$

$$E_q^-(k) = R_q x(k) - R_q \hat{x}^-(k). \quad (17)$$

Thus,

$$E_q^+(k+1) = R_q \hat{x}^+(k+1) - R_q x(k+1), \quad (18)$$

$$E_q^-(k+1) = R_q x(k+1) - R_q \hat{x}^-(k+1). \quad (19)$$

From (6) and (12), the dynamics of E_q^+ are given by:

$$E_q^+(k+1) = F_q E_q^+(k) + \gamma_q^+, \quad (20)$$

where

$$F_q = R_q (A_q - L_q C_q) R_q^{-1}, \quad (21)$$

and

$$\gamma_q^+(k) = |R_q|\bar{d}(k) - R_qd(k). \quad (22)$$

Similarly, the dynamics of E_q^- are given by:

$$\begin{aligned} E_q^-(k+1) &= R_qx(k+1) - R_q\hat{x}^-(k+1) \\ &= F_qE_q^-(k) + \gamma_q^-, \end{aligned} \quad (23)$$

with

$$\gamma_q^-(k) = |R_q|\bar{d}(k) + R_qd(k). \quad (24)$$

Bearing in mind Lemma 1, then

$$-R_q^+\bar{d}(k) - R_q^-\bar{d}(k) \leq R_qd(k) \leq R_q^+\bar{d}(k) + R_q^-\bar{d}(k). \quad (25)$$

By using (25) we have $R_qd(k) + |R_q|\bar{d}(k) \geq 0$, $|R_q|\bar{d}(k) - R_qd(k) \geq 0$, then we deduce that $\gamma_q^-(k) \geq 0$ and $\gamma_q^+(k) \geq 0, \forall k \geq 0$.

Moreover we have $\underline{x}(0) \leq x(0) \leq \bar{x}(0)$, then $E_q^-(0) = R_qx(0) - (R_q^+\underline{x}(0) - R_q^-\bar{x}(0))$ and $E_q^+(0) = R_q^+\bar{x}(0) - R_q^-\underline{x}(0) - R_qx(0)$ are nonnegative. As $F_q = R_q(A_q - L_qC_q)R_q^{-1}$ is nonnegative due to Assumption 4, we deduce that $E_q^-(k) \geq 0$ and $E_q^+(k) \geq 0, \forall k \geq 0$.

Consequently, we obtain

$$R_q\hat{x}^-(k) \leq R_qx(k) \leq R_q\hat{x}^+(k). \quad (26)$$

From (26) and (15), it can be verified that

$$\underline{x}(k) \leq x(k) \leq \bar{x}(k).$$

□

Remark 2. *The second approach based on changes of coordinates is general since it is always possible to transform any real square matrix into a nonnegative form. The existence of such a transformation is not a strong assumption. For instance, it has been shown in (Efimov et al., 2013) that there always exists an invertible matrix P such that in the coordinates $z(k) = Px(k)$, the matrix $E = P(A - LC)P^{-1}$ is nonnegative. In addition, it has been shown in (Mazenc et al., 2014) that based on the Jordan canonical form, it is always possible to transform any square constant matrix into a nonnegative form with a constant or a time-varying transformation.*

Remark 3. *The cooperativity property has motivated the need for state transformation. The interest of the second structure proposed above is that even by using changes of coordinates $z(k) = R_qx(k)$, the framer (12)-(13) is designed in the original coordinates "x" (i.e. $\hat{x}^+(k+1) = (A_q - L_qC_q)\hat{x}^+(k) + \dots$) instead of in the basis "z" (i.e. $\hat{x}^+(k+1) = R_q(A_q - L_qC_q)R_q^{-1}\hat{x}^+(k) + \dots$) as in (Guo & Zhu, 2017). This makes the stability analysis simpler and allows one to avoid jumping of the framer state and a hybrid behavior in the basis "z".*

3.3. Optimal gain computation

We devote this section to the computation of gains L_q to ensure stability properties which turn framers (8) and (15) into interval observers. Notice that these gains decide also the tightness of the interval width. Hence, the goal is not only to ensure the stability of the ultimate-bound but also to improve the accuracy of the proposed framers. The idea is based on H_∞ design. In other words, we are interested in computing observer gains L_q which minimize the following cost function

$$\begin{aligned} & \underset{L_q \in \mathbb{R}^{n \times p}}{\text{minimize}} && \gamma^2, \quad q = 1, \dots, N \\ & \text{subject to} && \frac{\|e\|_2^2}{\|\delta\|_2^2} \leq \gamma^2. \end{aligned} \quad (27)$$

With e is the estimation error depending on the interval width $\bar{x}(k) - \underline{x}(k)$ and δ is an input which takes into account the bound of the disturbances d . Where γ is a positive real number. The effect of the known bound of the uncertainties δ on the estimation error e is reduced by the observer gain matrices L_q . According to first or second framer design ((8) or (15)), we define e and δ , respectively, as follows

- (1) *In the case of first framer design.* Given the framer (8) for the system (6). We define the estimation error $e(k) = \bar{x}(k) - \underline{x}(k)$ and the input $\delta(k) = \bar{d}(k)$. Then, from (9) and (10), we obtain immediately

$$e(k+1) = (A_q - L_q C_q)e(k) + T_n \delta(k),$$

with $T_n = 2I_n$.

- (2) *In the case of second framer design.* Given the framer (12)-(15) for the system (6). We define the estimation error $e(k) = R_q^{-1}|S_q|^{-1}(\bar{x}(k) - \underline{x}(k))$ and the input $\delta(k) = R_q^{-1}|R_q|\bar{d}(k)$, with R_q, S_q are constant changes of coordinates in (14). Then, from (15) we have

$$\begin{aligned} e(k+1) &= R_q^{-1}|S_q|^{-1}(S_q^+ R_q + S_q^- R_q) (\hat{x}^+(k+1) - \hat{x}^-(k+1)) \\ &= R_q^{-1}|S_q|^{-1}|S_q|R_q (\hat{x}^+(k+1) - \hat{x}^-(k+1)) \\ &= \hat{x}^+(k+1) - \hat{x}^-(k+1) \end{aligned}$$

Due to (18)-(19), (20) and (23), one can prove through simple calculations that

$$\begin{aligned} e(k+1) &= R_q^{-1} (E_q^+(k+1) + E_q^-(k+1)) \\ &= (A_q - L_q C_q) R_q^{-1} (E_q^+(k) + E_q^-(k)) + 2R_q^{-1}(R_q^+ \bar{d}(k) + R_q^- \bar{d}(k)) \\ &= (A_q - L_q C_q) R_q^{-1} R_q e(k) + 2R_q^{-1}|R_q|\bar{d}(k) \\ &= (A_q - L_q C_q)e(k) + T_n \delta(k), \end{aligned}$$

with $T_n = 2I_n$.

Remark 4. *In both two cases, with the suitable choices of e and δ , we always have*

$$e(k+1) = (A_q - L_q C_q)e(k) + T_n \delta(k), \quad (28)$$

with $T_n = 2I_n$.

To simplify exposition, from now on we replace $e(k)$ and $\delta(k)$ by the subscripts e_k and δ_k , respectively. Note that the optimization problem (27) can be reformulated under LMI form through the bounded-real lemma (Boyd et al, 1994) to the linear switched dynamics (28). This leads to the following inequality

$$e_{k+1}^T P e_{k+1} - e_k^T P e_k \leq -e_k^T e_k + \gamma^2 \delta_k^T \delta_k, \quad (29)$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix.

We are ready to propose the theorem as guidelines for selecting gains L_q for (8) and (15).

Theorem 3 (Gain computaion for (8) and (15)). *If Assumptions 1-2 are satisfied and if there exists a symmetric positive definite matrix P such that*

$$\begin{bmatrix} -P + I_n & 0_{n \times n} & A_q^T P - C_q^T U_q^T \\ 0_{n \times n} & -\gamma^2 I_n & 2P \\ P A_q - U_q C_q & 2P & -P \end{bmatrix} \preceq 0, \quad (30)$$

with $U_q = P L_q$, the following statements hold true:

- (i) *If Assumption 3 is satisfied, the framers proposed in (8) become interval observers for (6).*
- (ii) *If Assumption 4 is satisfied, the framers proposed in (15) become interval observers for (6).*

Moreover, the optimal observer gain matrix,

$$L_q = P^{-1} U_q \quad (31)$$

is computed via the solution of the following constrained minimization problem

$$\begin{aligned} & \underset{P, U_q}{\text{minimize}} \quad \gamma^2, \quad q = 1, \dots, N \\ & \text{subject to} \quad (30). \end{aligned} \quad (32)$$

Proof. From (28) and (29), one can obtain

$$((A_q - L_q C_q) e_k + T_n \delta_k)^T P ((A_q - L_q C_q) e_k + T_n \delta_k) - e_k^T P e_k + e_k^T e_k - \gamma^2 \delta_k^T \delta_k \leq 0. \quad (33)$$

Inequality (33) is satisfied if and only if there exist P and γ such that

$$\begin{bmatrix} (A_q - L_q C_q)^T P (A_q - L_q C_q) - P + I_n & (A_q - L_q C_q)^T P T_n \\ T_n^T P (A_q - L_q C_q) & T_n^T P T_n - \gamma^2 I_n \end{bmatrix} \preceq 0. \quad (34)$$

To get LMI version for (34), some intermediate steps are necessary. We can rewrite (34) as

$$\begin{bmatrix} -P + I_n & 0_{n \times n} \\ 0_{n \times n} & -\gamma^2 I_n \end{bmatrix} + \begin{bmatrix} (A_q - L_q C_q)^T P \\ T_n^T P \end{bmatrix} P^{-1} \begin{bmatrix} P (A_q - L_q C_q) & P T_n \end{bmatrix} \preceq 0. \quad (35)$$

Then, based on the Schur complement introduced in Lemma 2, we can verify that

$$\begin{bmatrix} -P + I_n & 0_{n \times n} & A_q^T P - C_q^T L_q^T P \\ 0_{n \times n} & -\gamma^2 I_n & 2P \\ PA_q - PL_q C_q & 2P & -P \end{bmatrix} \preceq 0. \quad (36)$$

Hence,

$$\begin{bmatrix} -P + I_n & 0_{n \times n} & A_q^T P - C_q^T U_q^T \\ 0_{n \times n} & -\gamma^2 I_n & 2P \\ PA_q - U_q C_q & 2P & -P \end{bmatrix} \preceq 0, \quad (37)$$

with $U_q = PL_q$. \square

What is important to note here is that this methodology can be applied for the two different proposed framer designs. The same LMI can be used to compute optimal observer gains based on H_∞ formalism under suitable choices of e and δ .

4. Comparative simulations

In this section, we compare our results with the ones introduced in (Marouani et al., 2018) to highlight the contributions of this note. Consider the discrete-time linear switched system subject to disturbances of the form (6):

$$\begin{cases} x(k+1) = A_q x(k) + B_q u(k) + d(k) \\ y(k) = C_q x(k) \end{cases}, \quad q \in \overline{1, N}, \quad N \in \mathbb{N},$$

where $N = 3$,

$$A_1 = \begin{pmatrix} 0.2 & -0.5 \\ 0 & 0.2 \end{pmatrix}, A_2 = \begin{pmatrix} 0.3 & -2 \\ 0 & 0.6 \end{pmatrix}, A_3 = \begin{pmatrix} 0.5 & -1.1 \\ 0 & 0.16 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, B_2 = \begin{pmatrix} 6 \\ 1 \end{pmatrix}, B_3 = \begin{pmatrix} -2 \\ 2 \end{pmatrix},$$

$$C_1 = (0.2 \quad 0.8), C_2 = (1 \quad 0), C_3 = (0.1 \quad 1),$$

$d(k) = 0.1 \begin{pmatrix} \sin(0.5k) \\ \cos(0.5k) \end{pmatrix}$ is the bounded disturbances such that $-\bar{d} \leq d(k) \leq \bar{d}$ with $\bar{d} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$.

4.1. Optimized gains

The solution which minimizes $\alpha = \gamma^2$, obtained using the package CVX (Grant et al., 2014) is given by

$$L_1 = \begin{pmatrix} -0.026 \\ 0.0914 \end{pmatrix}, L_2 = \begin{pmatrix} 0.5416 \\ -0.0758 \end{pmatrix}, L_3 = \begin{pmatrix} -0.857 \\ 0.1028 \end{pmatrix},$$

$$P = \begin{pmatrix} 2.3947 & 1.8427 \\ 1.8427 & 14.8912 \end{pmatrix}, \alpha = 99.6689.$$

We verify that the matrices $A_q - L_q C_q$ are not nonnegative for all $q \in \overline{1, 3}$ so changes of coordinates are required. As discussed above, it is difficult to find a common R such that $R(A_q - L_q C_q)R^{-1}$ are all nonnegative for all $q \in \overline{1, 3}$. Consequently, we propose for the next changes of coordinates R_q satisfying that the matrices $R_q(A_q - L_q C_q)R_q^{-1}$ are nonnegative for all $q \in \overline{1, 3}$

$$R_1 = \begin{pmatrix} 0.0901 & 0.6930 \\ -0.0901 & 0.3070 \end{pmatrix}, R_2 = \begin{pmatrix} -0.2372 & -0.8173 \\ 0.2372 & 1.8173 \end{pmatrix},$$

$$R_3 = \begin{pmatrix} 0.0191 & 0.9913 \\ -0.0191 & 0.0087 \end{pmatrix}, R_1^{-1} = \begin{pmatrix} 3.4083 & -7.6972 \\ 1 & 1 \end{pmatrix},$$

$$R_2^{-1} = \begin{pmatrix} -7.661 & -3.4453 \\ 1 & 1 \end{pmatrix}, R_3^{-1} = \begin{pmatrix} 0.4558 & -51.8598 \\ 1 & 1 \end{pmatrix}.$$

For simulations, we use the initial conditions $x_0 = [1, 2]^T$, $\bar{x}_0 = (1.5, 2.5)^T$, $\underline{x}_0 = (0.5, 1.5)^T$. The results of simulation of the optimized observer are depicted in Figures 1-2 for both coordinates where solid lines present the state and dashed lines present the estimated bounds. Figure 1 illustrates the result in the case where there is no disturbances. We see clearly that the interval length converges to zero. Figure 2 shows that in the case where the system is affected by additive disturbances, the interval observer still provides the solutions with bounds. The switching signal between the three subsystems is plotted in Figure 3.

4.2. Comparisons

In (Marouani et al., 2018) the following LMI, which turns framers (8) and (15) into interval observers without considering the tightness problem of the interval widths, is proposed

$$\begin{bmatrix} P & U_q C_q \\ C_q^T U_q^T & \frac{\delta_q}{1 + \delta_q} P - A_q^T P A_q + A_q^T U_q C_q + (A_q^T U_q C_q)^T \end{bmatrix} \succ 0, q \in \overline{1, N}, N \in \mathbb{N},$$
(38)

with P, U_q defined in Theorem 3 and $\delta_q > 0$. Using the Yalmip toolbox (Löfberg, 2004), the gains obtained in (Marouani et al., 2018) are given by

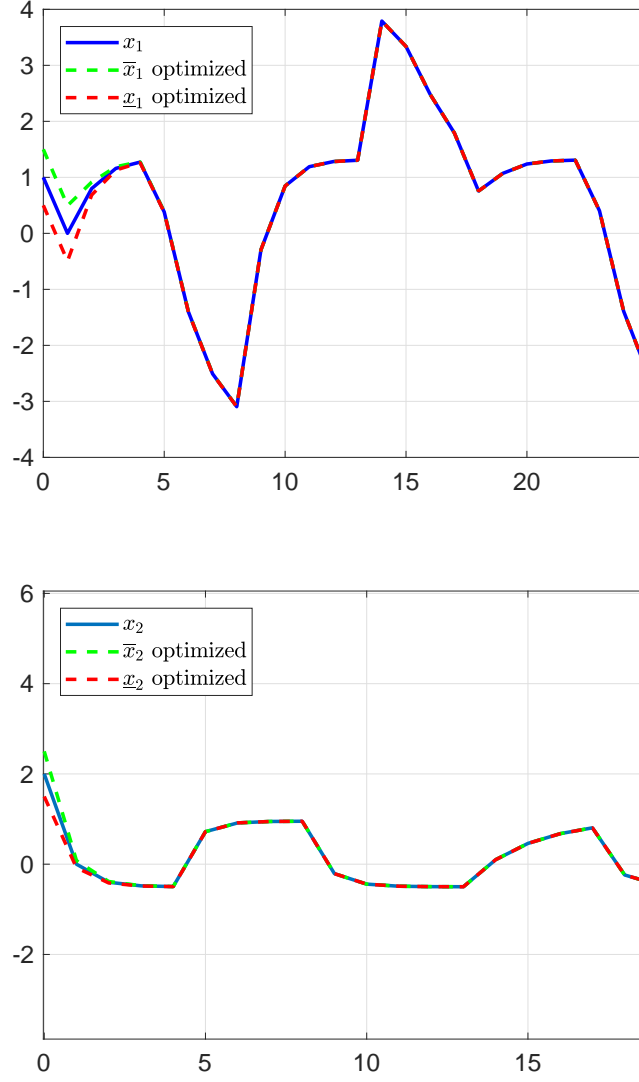


Figure 1. State and estimated bounds without disturbances

$$L_1 = \begin{pmatrix} -0.0954 \\ 0.0388 \end{pmatrix}, L_2 = \begin{pmatrix} 0.7253 \\ -0.1286 \end{pmatrix}, L_3 = \begin{pmatrix} -0.8067 \\ 0.166 \end{pmatrix},$$

$$P = \begin{pmatrix} 0.1818 & 0.23 \\ 0.23 & 1.0198 \end{pmatrix}, \delta_q = 1.588 \quad \forall q \in \overline{1, 3}.$$

We verify that the matrices $A_q - L_q C_q$ are not nonnegative for all $q \in \overline{1, 3}$ so changes of coordinates are required. Therefore for the solution of the LMI (38), we propose changes of coordinates R_q such that the matrices $R_q(A_q - L_q C_q)R_q^{-1}$ are nonnegative for all $q \in \overline{1, 3}$ as follows

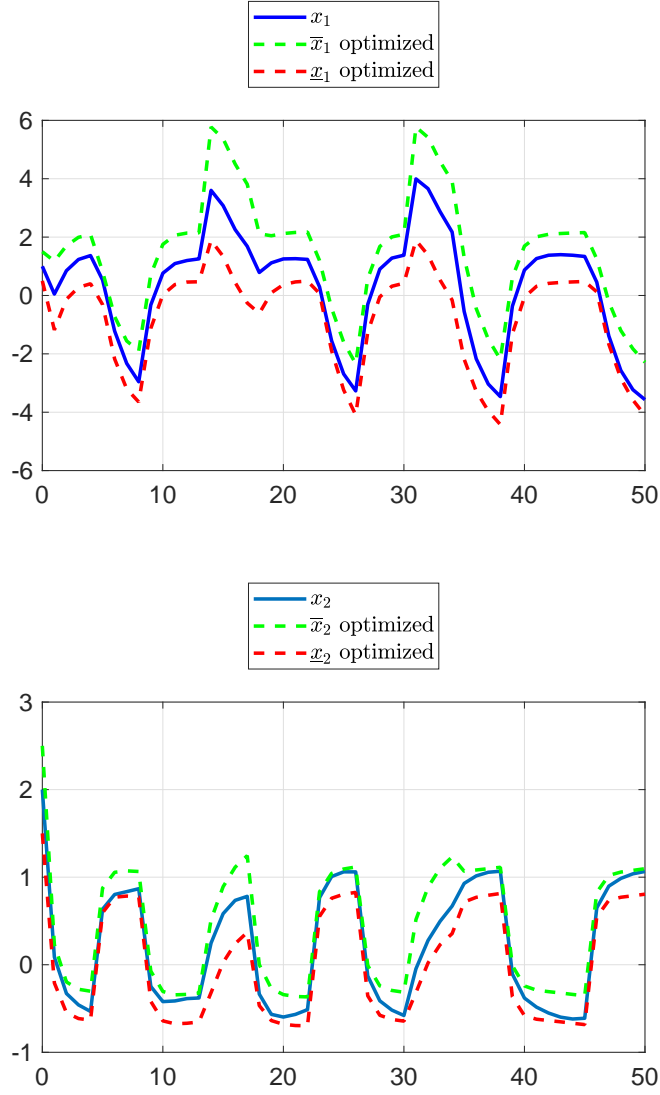


Figure 2. State and estimated bounds with the disturbances

$$\begin{aligned}
 R_1 &= \begin{pmatrix} 0.062 & 0.7002 \\ -0.062 & 0.2998 \end{pmatrix}, R_2 = \begin{pmatrix} -0.8664 & -2.9526 \\ 0.8664 & 3.9526 \end{pmatrix}, \\
 R_3 &= \begin{pmatrix} 0.0275 & 0.9864 \\ -0.0275 & 0.0136 \end{pmatrix}, R_1^{-1} = \begin{pmatrix} 4.8366 & -11.2966 \\ 1 & 1 \end{pmatrix}, \\
 R_2^{-1} &= \begin{pmatrix} -4.562 & -3.4078 \\ 1 & 1 \end{pmatrix}, R_3^{-1} = \begin{pmatrix} 0.4932 & -35.8415 \\ 1 & 1 \end{pmatrix}.
 \end{aligned}$$

With the same initial conditions introduced in Section 4.1, the simulation of the optimized bounds and of the bounds given by (Marouani et al., 2018) are depicted in the same Figure 4 for the purpose of comparison. Both coordinates are drawn where solid lines present the state and dashed lines present the estimated bounds.

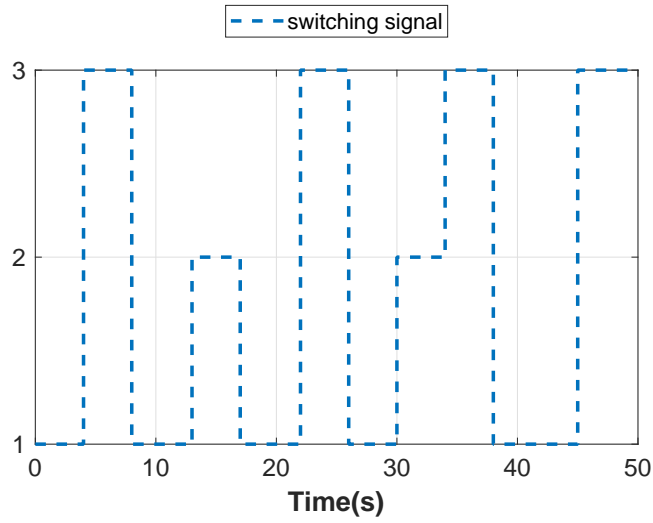
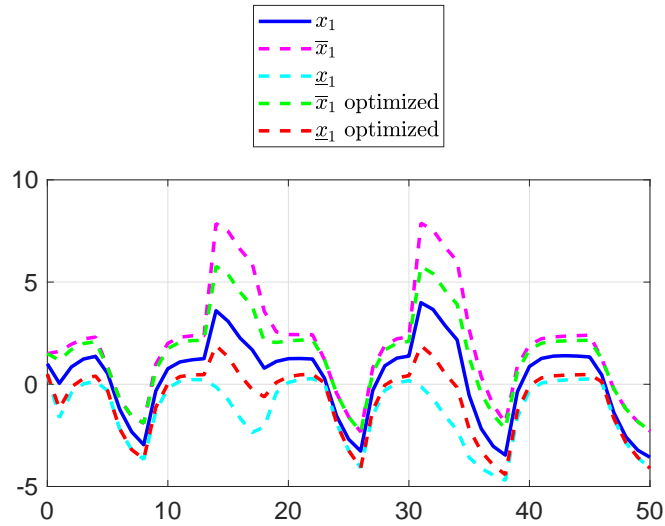


Figure 3. Switching signal



The simulation results show that the interval width obtained by our H_∞ design is tighter than the approach proposed in (Marouani et al., 2018). The accuracy of framer is clearly improved.

5. Conclusion

In this paper, two techniques to design interval observers for a class of discrete-time linear switched systems in the presence of additive disturbances are proposed. The assumptions given in the first one are not always feasible. Therefore, a second approach based on changes of coordinates is proposed to relax the condition of nonnegativeness of $A_q - L_q C_q$, $q = \overline{1, N}$. In this context, two copies of classical observers associated with suitably selected initial conditions are reformulated in the base "x". For improvement

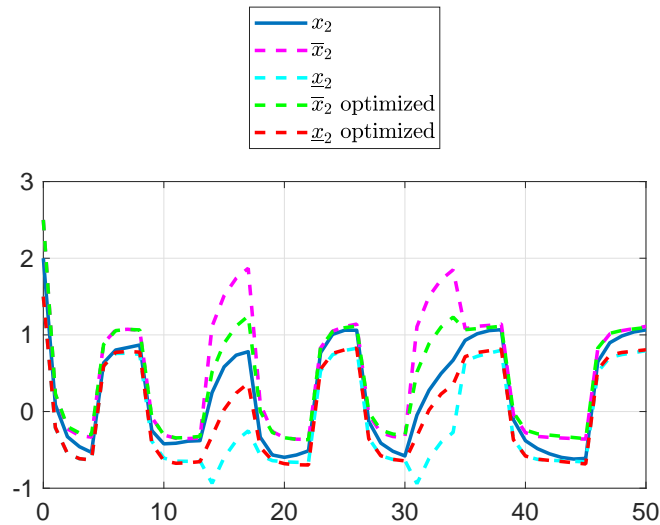


Figure 4. State and estimated bounds

of performance for estimation, the present note has also introduced H_∞ method to estimate optimally the state. The effectiveness of the method is confirmed through comparative simulations. Observer gains can be computed in term of LMIs. Extensions to switched systems with unknown switching instants are expected for further studies.

References

- V. Alcaraz-Gonzalez and V. Gonzalez-Alvarez, Robust nonlinear observers for bioprocesses: application to wastewater treatment, *Dyn. and ctrl. of chem. and bio. processes*, Lecture notes in control and information sciences, Berlin, Springer-Verlag, pp. 119-164, 2007.
- O. Bernard and J.L. Gouzé, Closed loop observers bundle for uncertain biotechnical models, *Journal of Process Control*, Vol 14, pp. 765-774, 2004.
- S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, Philadelphia, PA, June 1994.
- M.S. Branicky, Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control*, Vol 43(4), pp. 475-482, 1998.
- C. Briat, M. Khammash, Simple interval observers for linear impulsive systems with applications to sampled-data and switched systems, 20th IFAC WC 2017, Toulouse, France, pp. 5079-5084, 2017.
- T.N. Dinh, F. Mazenc, S.I. Niculescu, Interval observer composed of observers for nonlinear systems. In 13th European Control Conference, Strasbourg, France, pp. 660-665, 2014.
- T.N. Dinh and H. Ito, Interval Observers for Continuous-time Bilinear Systems with Discrete-time Outputs, 15th European Control Conference, Aalborg, Denmark, pp. 1418-1423, 2016.
- D. Efimov, L. Fridman, T. Raïssi, Ali Zolghadri, Ramatou Seydou, Interval Estimation for LPV Systems Applying High Order Sliding Mode Techniques. *Automatica*, 2012, Vol 48, pp. 2365-2371.
- D. Efimov, W. Perruquetti, T. Raïssi, A. Zolghadri, On Interval Observers for Time-Varying Discrete-Time Systems. *IEEE Transactions on Automatic Control*, Vol 58(12), pp. 3218-3224, 2013.
- H. Ethabet, T. Raïssi, M. Amairi, M. Aoun, Interval observers design for continuous-time

- linear switched systems, 20th IFAC WC 2017, Toulouse, France, pp. 6259-6264, 2017.
- H. Ethabet, D. Rabehi, D. Efimov, T. Raïssi, Interval estimation for continuous-time switched linear systems. *Automatica*, Vol. 90, pp. 230-238, 2018.
- G. Goffaux, A. Vande Wouwer and O. Bernard, Improving continuous discrete interval observers with application to microalgae-based bioprocesses, *Journal of Process Control*, Vol 19, pp. 1182-1190, 2009.
- J.L. Gouzé, A. Rapaport and M.Z. Hadj-Sadok, Interval observers for uncertain biological systems, *Ecological modelling*, Vol 133(1-2), pp. 45-56, 2000.
- M. Grant, S. Boyd, CVX: Matlab software for disciplined convex programming, version 2.1, March 2014.
- S. Guo and F. Zhu, Interval observer design for discrete-time switched system, 20th IFAC WC 2017, Toulouse, France, pp. 5073-5078, 2017.
- Z. He, W. Xie, Control of non-linear switched systems with average dwell time: interval observer-based framework. *IET Control Theory and Application*, Vol 10(1): 10-16, 2016.
- S. Ifqir, N.A. Oufroukh, D. Ichalal, S. Mammam, Switched interval observer for uncertain continuous-time systems. In *Proc. 20th IFAC WC 2017*.
- H. Ito, T.N. Dinh, Interval Observers for Global Feedback Control of Nonlinear Systems with Robustness with respect to Disturbances, *European Journal of Control*, Vol 39, pp. 68-77, 2018.
- H. Lin, P.J. Antsaklis, Stability and stabilizability of switched linear systems: a survey of recent results. *IEEE Transactions on Automatic Control*, Vol 54, pp. 308-322, 2009.
- J. Löfberg, Yalmip: A toolbox for modeling and optimization in MATLAB, in *Proc. CACSD Conf.*, Taipei, Taiwan, 2004.
- N. Loukkas, J. J. Martinez, N. Meslem, Set-membership observer design based on ellipsoidal invariant sets, 20th IFAC WC 2017, Toulouse, France, pp. 6471-6476, 2017.
- D.G. Luenberger, An Introduction to Observers, *IEEE Transactions on Automatic Control*, Vol 16(1), pp. 596-602, 1971.
- G. Marouani, T.N. Dinh, T. Raïssi and H. Messaoud, Interval observers design for discrete-time linear switched systems, 17th European Control Conference, Limassol, Cyprus, pp. 801-806, 2018.
- F. Mazenc and O. Bernard, Asymptotically stable interval observers for planar systems with complex poles, *IEEE Transactions on Automatic Control*, Vol 55, No.2, pp. 523-527, 2010.
- F. Mazenc and O. Bernard, Interval observers for linear time-invariant systems with disturbances, *Automatica*, Vol 47(1), pp. 140-147, 2011.
- F. Mazenc, T.N. Dinh and S.I. Niculescu, Robust interval observers and stabilization design for discrete-time systems with input and output, *Automatica*, Vol 49, pp. 3490-3497, 2013.
- F. Mazenc, T.N. Dinh, S.L. Niculescu, Interval observers for discrete-time systems, *International Journal of Robust and Nonlinear Control*, Vol 24(17), pp. 2867-2890, 2014.
- F. Mazenc and T.N. Dinh, Construction of interval observers for continuous-time systems with discrete measurements, *Automatica*, Vol 50(10), pp. 2555-2560, 2014.
- M. Moisan, O. Bernard and J.L. Gouzé, Near optimal interval observers bundle for uncertain bioreactor, *Automatica*, Vol 45, pp. 291-295, 2009.
- T. Raïssi, N. Ramdani and Y. Candau, Bounded error moving horizon state estimation for non-linear continuous time systems: application to a bioprocess system, *Journal of Process Control*, Vol 15, pp. 537-545, 2005.
- T. Raïssi, D. Efimov and A. Zolghadri, Interval state estimation for a class of nonlinear systems, *IEEE Transactions on Automatic Control*, Vol 57(1), pp. 260-265, 2012.
- Z. Wang, C.-C. Lim, Y. Shen. Interval observer design for uncertain discrete-time linear systems, *Systems & Control Letters*, Vol. 116, pp. 41-46, 2018.