Generating the Skew Normal random variable

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***Abstract*—In this paper, for the generating the Skew normal random variables, we propose a new method based on the combination of minimum and maximum of two independent normal random variables. The estimation of parameters using the maximum likelihood estimation and the methods of moments estimation method. A real data set has been considered to illustrate the practical utility of the paper.**

keyword: Skew normal distribution, maximum likelihood estimation, methods of moments estimation.

I. Introduction

The univariate Skew Normal distribution has been studied by Azzalini [1] , Azzalini [2] , Henze [3] and Pewsey [4], has been studied and generalized extensively. The skew normal distribution family is well known for modeling and analyzing skewed data. It is the distribution family that extends the normal distribution family by adding a shape parameter to regulate the skewness, which has the higher exibility in tting a real data where some skewness is present. The density function of the Skew Normal distribution with parameters (ξ, τ, θ) is given by

$$\frac{2}{\tau}\,\varphi\Big(\frac{x-\xi}{\tau}\Big)\,\Phi\Big(\theta\Big(\frac{x-\xi}{\tau}\Big)\Big) \qquad (1)$$

where ξ is location parameters , $\tau > 0$ is scale parameters , θ is asymmetry parameter and φ and Φ are, respectively, the density and cumulative distribution functions of the standard normal distribution $\mathcal{N}(0,1)$. In this paper we use the notation $\mathcal{SN}\big(\xi, \tau, \theta\big)$ to denote this distribution.

For the simulation of the $\mathcal{SN}\big(0,1,\theta\big)$ distribution, Henze [3], in his paper showed that, if U_1 and U_2 are identically and independently distributed $\mathcal{N}(0,1)$ random variables, then $\dfrac{\theta|U_1| + U_2}{\sqrt{1+\theta^2}}$ has the $\mathcal{SN}\big(0,1,\theta\big)$ distribution.

For the simulation of $\mathcal{SN}\big(0,1,\theta\big)$ distribution, recently Ghorbanzadeh et al [5], have developed a method, called the Min-Max method, which consists in taking the combination of minimum and maximum of two independent random variables distributed $\mathcal{N}(0,1)$. In this parpier, for the simulation of $\mathcal{SN}\big(\xi,\tau,\theta\big)$ distribution, we will use the Min-Max method.

II. Generating random variables by Min-Max method

Let $m = \sqrt{\dfrac{1+\theta^2}{2}}\,\xi$, U_1 and U_2 two independent and identically distributed $\mathcal{N}(m, \tau^2)$ random variables and $U = \max(U_1, U_2)$ and $V = \min(U_1, U_2)$. For simulation of the random variable $X \sim \mathcal{SN}\big(\xi,\tau,\theta\big)$, we take the combination of U and V. First note that:

- if $\theta = 0$, the density (1) becomes: $\dfrac{1}{\tau}\,\varphi\Big(\dfrac{x-\xi}{\tau}\Big)$, simply simulate $X \sim \mathcal{N}(\xi, \tau^2)$.
- if $\theta = -1$, the density (1) becomes:

$$\frac{2}{\tau}\,\varphi\Big(\frac{x-\xi}{\tau}\Big)\,\Big(1 - \Phi\Big(\frac{x-\xi}{\tau}\Big)\Big)$$

  we take $X = \min(X_1, X_2)$.
- if $\theta = 1$, the density (1) becomes:

$$\frac{2}{\tau}\,\varphi\Big(\frac{x-\xi}{\tau}\Big)\,\Phi\Big(\frac{x-\xi}{\tau}\Big)$$

  we take $X = \max(X_1, X_2)$.

where X_1 and X_2 are identically and independently distributed $\mathcal{N}(\xi, \tau^2)$ random variables.

For $\theta \notin \{-1, 0, 1\}$, note :

$$\lambda_1 = \frac{1+\theta}{\sqrt{2(1+\theta^2)}} \quad , \quad \lambda_2 = \frac{1-\theta}{\sqrt{2(1+\theta^2)}} \qquad (2)$$

We note that: $\lambda_1^2 + \lambda_2^2 = 1$. For simulation of the random variable $X \sim \mathcal{SN}\big(\xi,\tau,\theta\big)$, we take the combination of U and V in the form:

$$X = \lambda_1 U + \lambda_2 V \qquad (3)$$

Proposition. *The random variable X defined in the equation (3) has the Skew Normal distribution* $\mathcal{SN}\big(\xi,\tau,\theta\big)$.

Proof: The pair (U, V) has density:

$$f_{U,V}(u,v) = \frac{2}{\tau^2}\,\varphi\Big(\frac{u-m}{\tau}\Big)\,\varphi\Big(\frac{v-m}{\tau}\Big)\,\mathbb{1}_{\{v\le u\}}(u,v) \quad (4)$$

where $\mathbb{1}$ is the indicator function.

Consider the transformation: $x = \lambda_1 u + \lambda_2 v$, $y = \lambda_1 u$. The inverse transform is defined by: $u = \dfrac{y}{\lambda_1}$, $v = \dfrac{x-y}{\lambda_2}$ and the corresponding Jacobian is: $J = \dfrac{1}{|\lambda_1\lambda_2|}$. X density is defined by:

$$\begin{aligned} f(x) &= \frac{1}{|\lambda_1\lambda_2|}\int_\Delta f_{U,V}\Big(\frac{y}{\lambda_1}, \frac{x-y}{\lambda_2}\Big)\,dy \\ &= \frac{2}{\tau^2\,|\lambda_1\lambda_2|}\int_\Delta \varphi\Big(\frac{y-m\lambda_1}{\lambda_1\tau}\Big)\,\varphi\Big(\frac{y-x+m\lambda_2}{\lambda_2\tau}\Big)\,dy \end{aligned} \qquad (5)$$

where $\Delta = \{\dfrac{x-y}{\lambda_2} \le \dfrac{y}{\lambda_1}\}$.

For the density of the standard normal distribution $\mathcal{N}(0,1)$,

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we have the following classical property,

$$\varphi\Big(\frac{x-\mu_1}{\sigma_1}\Big)\varphi\Big(\frac{x-\mu_2}{\sigma_2}\Big) = \varphi\Bigg(\frac{\sqrt{\sigma_1^2+\sigma_2^2}}{\sigma_1\sigma_2}\Big(x-\frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}\Big)\Bigg)\varphi\Bigg(\frac{\mu_2-\mu_1}{\sqrt{\sigma_1^2+\sigma_2^2}}\Bigg) \tag{6}$$

By using relation (6), the relation (5) becomes

$$f(x) = \frac{2}{\tau^2\mid\lambda_1\lambda_2\mid}\varphi\Big(\frac{x-m(\lambda_1+\lambda_2)}{\tau}\Big) \times \int_\Delta \varphi\Bigg(\frac{1}{\lambda_1\lambda_2\tau}\Big(y-x\lambda_1^2-m\lambda_1\lambda_2(\lambda_2-\lambda_1)\Big)\Bigg)\,dy \tag{7}$$

Taking account of $m(\lambda_1+\lambda_2)=\xi$, we obtain,

$$f(x) = \frac{2}{\tau^2\mid\lambda_1\lambda_2\mid}\varphi\Big(\frac{x-\xi}{\tau}\Big) \times \int_\Delta \varphi\Bigg(\frac{1}{\lambda_1\lambda_2\tau}\Big(y-x\lambda_1^2-m\lambda_1\lambda_2(\lambda_2-\lambda_1)\Big)\Bigg)\,dy \tag{8}$$

For the domain Δ, we have the following three cases:

case 1 $\theta\in(-1,0)\cup(0,1)$, we have:

$$\mid\lambda_1\lambda_2\mid = \frac{1-\theta^2}{2(1+\theta^2)} \text{ and } \Delta=\{y\geq\frac{\lambda_1 x}{\lambda_1+\lambda_2}\}.$$

case 2 $\theta<-1$, we have: $\mid\lambda_1\lambda_2\mid = \frac{\theta^2-1}{2(1+\theta^2)}$ and

$$\Delta=\{y\geq\frac{\lambda_1 x}{\lambda_1+\lambda_2}\}$$

case 3 $\theta>1$, we have: $\mid\lambda_1\lambda_2\mid = \frac{\theta^2-1}{2(1+\theta^2)}$ and

$$\Delta=\{y\leq\frac{\lambda_1 x}{\lambda_1+\lambda_2}\}$$

Using equation (8) and the three cases above, we get the result. ∎

III. INFERENCE

Let $X_1,\ldots,X_n$ be a sample of size n from a $\mathcal{SN}(\xi,\tau,\theta)$ distribution.

A. Maximum likelihood Estimation

The log-likelihood is given by

$$\ell(\xi,\tau,\theta) = n\log 2 - n\log\tau + \sum_{i=1}^{n}\log\Bigg(\varphi\Big(\frac{x_i-\xi}{\tau}\Big)\Bigg) + \sum_{i=1}^{n}\log\Bigg(\Phi\Bigg(\theta\Big(\frac{x_i-\xi}{\tau}\Big)\Bigg)\Bigg)$$

The maximum likelihood estimators (MLE) of (ξ,τ,θ), denoted $(\hat{\xi},\hat{\tau},\hat{\theta})$, are the numerical solution of the system of equations :

$$\begin{cases} \tau^2 = \frac{1}{n}\sum_{i=1}^{n}\big(x_i-\xi\big)^2 \\ \tau\theta\sum_{i=1}^{n}\frac{\varphi\Big(\theta\big(\frac{x_i-\xi}{\tau}\big)\Big)}{\Phi\Big(\theta\big(\frac{x_i-\xi}{\tau}\big)\Big)} = \sum_{i=1}^{n}\big(x_i-\xi\big) \\ \sum_{i=1}^{n}\frac{\big(x_i-\xi\big)\,\varphi\Big(\theta\big(\frac{x_i-\xi}{\tau}\big)\Big)}{\Phi\Big(\theta\big(\frac{x_i-\xi}{\tau}\big)\Big)} = 0 \end{cases} \tag{9}$$

B. Methods of Moments Estimation

Let m_k be the centered moment of order k of data defined by

$$m_k = \frac{1}{n}\sum_{i=1}^{n}\big(x_i-\overline{x}_n\big)^k$$

where $\overline{x}_n = \frac{1}{n}\sum_{i=1}^{n}x_i$, and denote by γ_1 the skewness coefficient defined by $\gamma_1=\frac{m_3}{m_2^{3/2}}$.

The method of moment estimators (MME) of (ξ,τ,θ), denoted $(\tilde{\xi},\tilde{\tau},\tilde{\theta})$, are obtained by solving the set of three equations:

$$\begin{cases} \xi + a\tau\delta = \overline{x}_n & \text{(10a)} \\ \tau^2\big(1-a^2\delta^2\big) = m_2 & \text{(10b)} \\ \frac{ba^3\delta^3}{\big(1-a^2\delta^2\big)^{3/2}} = \gamma_1 & \text{(10c)} \end{cases}$$

where $a=\sqrt{\frac{2}{\pi}}$, $b=\frac{4-\pi}{2}$ and $\delta=\frac{\theta}{\sqrt{1+\theta^2}}$.

We note that the sign of δ is the same as the sign of γ_1. The equation (10c) admits the solution

$$\tilde{\delta} = sign(\gamma_1)\sqrt{\frac{|\gamma_1|^{2/3}}{a^2\big(b^{2/3}+|\gamma_1|^{2/3}\big)}} \tag{11}$$

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where $sign(\gamma_1)$ is the sign of γ_1. Consequently,

$$\begin{cases} \tilde{\theta} = \dfrac{\tilde{\delta}}{\sqrt{1-\tilde{\delta}^2}} \\ \tilde{\tau} = \sqrt{\dfrac{m_2}{1-a^2\tilde{\delta}^2}} \\ \tilde{\xi} = \overline{x}_n - a\tilde{\tau}\tilde{\delta} \end{cases} \qquad (12)$$

IV. Simulation results

In order to study the performance of the method, we simulated 100-samples of sizes $n = 5000$. For each sample we calculated the parameter estimators, MLE and MME, the Table I summarizes the results obtained for different values of (ξ, τ, θ).

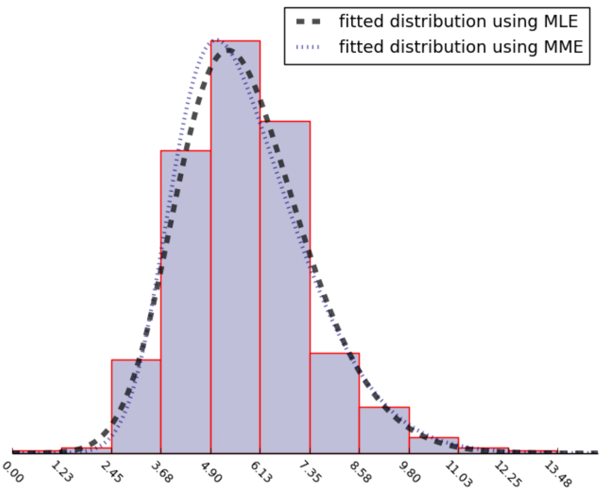
V. Application

In this section, we illustrate the use of the estimation procedures described in the previous section. The variable to be considered is the average length of stay for patients who are in hospital for acute care because of problems, hepatobiliary system and pancreas, and die for this cause. The sample, under study, corresponds to 1082 hospitals in 10 states of the United States. For more information see columns 4 in http://lib.stat.cmu.edu/data-expo/1997/ascii/p07.dat. The data were used by Elal-Olivero [6] by modeling with a model "alpha-Skew-Normal distribution".
Table II summarizes the results obtained by the two methods of estimation.

TABLE I: Estimators of (ξ, τ, θ) by the MLE and MME methods.

Parameter estimates	MLE	MME
ξ	4.1192	3.8856
τ	2.3006	2.4731
θ	2.0608	2.6286

By the MLE method, we get the same results as Elal-Olivero [6]. The results obtained by the MME method are similar as the results obtained by the MLE method.



Histogram corresponds to the data divided into 11 classes. The dashed line represent fitted distributions using the maximum likelihood estimators and the dotted line represent fitted distributions using the method of moment estimators

VI. Conclusion

The simulation method proposed in this paper is simpler to use than the three methods, the inverse transform method, the composition method and the acceptance-rejection method. The results of estimation of the parameters, by the two estimation methods, from the simulations are very satisfactory. For the application, we obtain the same estimation values as other author using the same data.

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TABLE II: Statistics of estimators of (ξ, τ, θ) for size $n = 5000$.

$\xi = -6, \tau = 8, \theta = -7$						
MLE	mean of $\hat{\xi}$	-6.0017	mean of $\hat{\tau}$	8.0045	mean of $\hat{\theta}$	-7.2612
	std of $\hat{\xi}$	0.2109	std of $\hat{\tau}$	0.2930	std of $\hat{\theta}$	1.2125
MME	mean of $\tilde{\xi}$	-6.1489	mean of $\tilde{\tau}$	7.9178	mean of $\tilde{\theta}$	-7.5284
	std of $\tilde{\xi}$	0.8805	std of $\tilde{\tau}$	0.4288	std of $\tilde{\theta}$	2.8694
$\xi = 2, \tau = 5, \theta = 3$						
MLE	mean of $\hat{\xi}$	2.0025	mean of $\hat{\tau}$	4.9891	mean of $\hat{\theta}$	3.0745
	std of $\hat{\xi}$	0.2396	std of $\hat{\tau}$	0.2382	std of $\hat{\theta}$	0.5178
MME	mean of $\tilde{\xi}$	2.0054	mean of $\tilde{\tau}$	5.0546	mean of $\tilde{\theta}$	3.2742
	std of $\tilde{\xi}$	0.0788	std of $\tilde{\tau}$	0.0795	std of $\tilde{\theta}$	0.1930

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