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SFLM: A mix of a Functional Linear Model and of a Spatial Autoregressive Model for spatially correlated functional data

Tingting Huang¹, Gilbert Saporta², Huiwen Wang¹, Shanshan Wang¹

¹ Beihang University, Beijing, China

² CNAM, Paris, France

Outline

1. Introduction
2. From regression integral to the functional linear model (FLM)
3. The spatial functional linear model (SFLM)
4. Simulation study
5. Application to weather data
6. Conclusions and perspectives

1. Introduction

- Notations:
 - \mathbf{y} a vector of n observations of a response variable
 - $\mathbf{x}(t)$ a set of n time functions (or curves) observed on $[0, T]$: the predictors
- The functional linear model (FLM) is an extension of multiple regression

$$\mathbf{y} = \alpha \mathbf{1} + \int_0^T \beta(t) \mathbf{x}(t) dt + \boldsymbol{\varepsilon}$$

- FLM usually assumes that the observations are independent. This is not the case for spatial or network data

- We propose here a simple combination of the Functional Linear Model and of the Spatial Autoregressive (SAR) Model to deal with spatially correlated data
- n spatially distributed data on a regular or not regular grid
- The SAR model commonly used in econometrics states that

$$\mathbf{y} = \rho \mathbf{W} \mathbf{y} + \mathbf{X} \boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

\mathbf{X} is a matrix of p predictors

\mathbf{W} is a spatial (or neighbouring) weight matrix. Rows sum to 1 and diagonal elements are 0.

ρ reflects the strength of spatial dependence.

SFLM : a linear regression with functional covariates and spatially correlated responses

$$\mathbf{y} = \alpha \mathbf{1} + \rho \mathbf{W} \mathbf{y} + \int_0^1 \beta(t) \mathbf{x}(t) dt + \boldsymbol{\varepsilon}$$

SFLM handles dependent responses via the autoregressive term $\rho \mathbf{W} \mathbf{y}$

Related works:

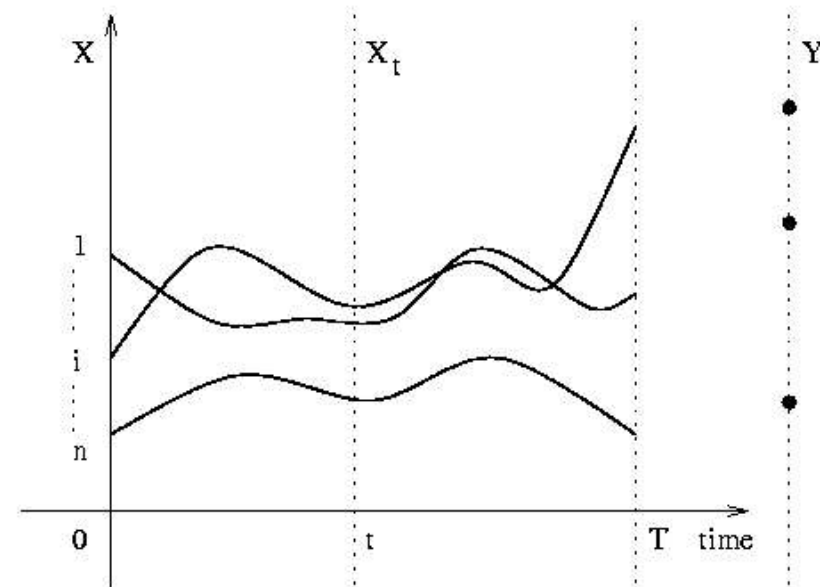
Pineda, W. and Giraldo, R. 2016., Aguilera-Morillo, M.C. et al, 2017, Giraldo et al. 2017

2. From regression integral to the functional linear model (FLM)

R.A.Fisher « The Influence of Rainfall on the Yield of Wheat at Rothamsted » Philosophical Transactions of the Royal Society, B: 213: 89-142 (1924)

Y = amount of crop

X_t = temperature curves



Disregarding, then, both the arithmetical and the statistical difficulties, which a direct attack on the problem would encounter, we may recognise that whereas with q subdivisions of the year, the linear regression equations of the wheat crop upon the rainfall would be of the form

$$\bar{w} = c + a_1 r_1 + a_2 r_2 + \dots + a_q r_q$$

where r_1, r_2, \dots, r_q are the quantities of rain in the several intervals of time, and a_1, \dots, a_q are the regression coefficients, so if infinitely small subdivisions of time were taken, we should replace the linear regression function by a *regression integral* of the form

$$\bar{w} = c + \int_0^T ar dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (III)$$

where $r dt$ is the rain falling in the element of time dt ; the integral is taken over the whole period concerned, and a is a *continuous* function of the time t , which it is our object to evaluate from the statistical data.

The functional linear model states that :

$$Y = \int_0^T \beta(t) X(t) dt + \varepsilon \text{ or } Y = \alpha + \int_0^T \beta(t) X(t) dt + \varepsilon$$

See Ramsay, J.O. & Silverman, B.W. (1997); Cardot et al. (1999); Cai and Hall (2006); Hall and Horowitz (2007) etc.

Minimizing $E\left(Y - \int_0^T \beta(t) X_t dt\right)^2$

leads to **Wiener-Hopf** equations: $\text{cov}(X_t, Y) = \int_0^T C(t, s) \beta(s) ds$

An ill posed problem

- Picard's theorem states that $\beta(t)$ is unique if and only if: $\sum_{i=1}^{\infty} \frac{c_i^2}{\lambda_i^2} < \infty$
where the λ_i are the eigenvalues of the Karhunen-Loève expansion of $X(t)$ and the c_i the covariances between Y and the functional principal components

$$X(t) = \sum_{i=1}^{\infty} f_i(t) \xi_i$$

$$c_i = \text{cov}(Y, \xi_i) = \text{cov}(Y, \int_0^T f_i(t) X(t) dt) = \int_0^T E(X(t)Y) f_i(t) dt$$

- Generally not true...especially when n is finite

Constrained solutions are needed. Several solutions:

- “roughness penalty” bounds on the integral of $(\beta'')^2$
(cf Green & Silverman 1994, Ramsay & Silverman 1997)

- Projection onto a finite dimensional subspace

PCR consists in a regression onto the m first principal components
(truncated KL expansion)

Extensions of FLM have been studied to address specific problems:

- Escabias, M., Aguilera A.M. and Valderrama M.J. (2004) discussed principal component estimation of functional logistic regression
- Aneiros-Prez and Vieu (2006) constructed a semi-functional partial linear model
- Ferraty et al. (2013) generalized the FLM to functional projection pursuit regression that allows for more interpretability
- Liu et al. (2017) presented a functional linear mixed model

3. The spatial functional linear model (SFLM)

$$\mathbf{y} = \alpha \mathbf{1} + \rho \mathbf{W} \mathbf{y} + \int_0^1 \beta(t) \mathbf{x}(t) dt + \boldsymbol{\varepsilon}$$

- The spatial weight matrix \mathbf{W} is exogenous. It is constructed according to distances between units under different contexts. For the case of geographic location, \mathbf{W} is formed according to adjacent relation or nearest k neighbors in terms of Euclidean distance or other metric.
- SFLM has merits of FLM and SAR simultaneously. It reduces to the classical FLM when $\rho = 0$

- A reformulation shows that the error terms are not independent:

$$\mathbf{y} = (\mathbf{I}_n - \rho \mathbf{W})^{-1} \alpha \mathbf{1} + (\mathbf{I}_n - \rho \mathbf{W})^{-1} \int_0^1 \beta(t) \mathbf{x}(t) dt + (\mathbf{I}_n - \rho \mathbf{W})^{-1} \boldsymbol{\varepsilon}$$

maximum likelihood should be used instead of OLS

- The intercept α , the spatial autocorrelation parameter ρ , the slope function $\beta(t)$, and the variance of the error term σ^2 are estimated by maximum likelihood combined with the basis expansion in terms of truncated functional PCA.
- Karhunen-Loeve expansion of $X(t)$ and basis expansion of $\beta(t)$

$$\mathbf{x}(t) = \sum_{j=1}^{\infty} \mathbf{a}_j \varphi_j(t) \quad \mathbf{a}_j = \int_0^1 \mathbf{x}(t) \varphi_j(t) dt$$

$$\beta(t) = \sum_{j=1}^{\infty} b_j \varphi_j(t) \quad b_j = \int_0^1 \beta(t) \varphi_j(t) dt$$

- SFLLM equation with truncated expansion

$$\mathbf{y} \simeq \alpha \mathbf{1} + \rho \mathbf{W} \mathbf{y} + \sum_{j=1}^m \mathbf{a}_j b_j + \boldsymbol{\varepsilon}$$

$$\mathbf{Z} = (\mathbf{1} \mid \mathbf{A}) \quad \boldsymbol{\delta} = \begin{pmatrix} \alpha \\ \mathbf{b} \end{pmatrix}$$

$$\mathbf{y} \simeq \rho \mathbf{W} \mathbf{y} + \mathbf{Z} \boldsymbol{\delta} + \boldsymbol{\varepsilon}$$

One gets the log-likelihood:

$\mathbf{e} = \mathbf{y} - \rho \mathbf{W} \mathbf{y} - \mathbf{Z} \boldsymbol{\delta}$ normally distributed with variance $\sigma^2 \mathbf{I}$

$$\ln L(\rho, \boldsymbol{\delta}, \sigma^2) = -\frac{n}{2} \ln(2\pi\sigma^2) + \ln |\mathbf{I} - \rho \mathbf{W}| - \frac{\mathbf{e}'\mathbf{e}}{2\sigma^2}$$

Hence the estimation of ρ and $\beta(t)$

4. Simulation

- Three regular grids of $n = \{10 \times 30; 20 \times 25; 30 \times 30\}$
- Each point has 2,3, or 4 neighbours, hence the matrix \mathbf{W}
- Same process as in Hall, P. and Horowitz, J. L. (2007)

$$\mathbf{y} = (\mathbf{I}_n - \rho \mathbf{W})^{-1} \left(\int_0^1 \mathbf{x}(t) \beta(t) dt + 0.5 \boldsymbol{\epsilon} \right), \quad \epsilon_i \sim N[0, 1]$$

- Three values for ρ : 0; 0.5; 0.8

- The functional predictor $x(t) = (x_1(t); x_2(t); \dots; x_n(t))$ is produced with independent functions

$$x_i(t) = \sum_{j=1}^{50} a_j Z_j \varphi_j(t),$$

where $a_j = (-1)^{j+1} j^{-\gamma/2}$ with $\gamma = 1.1$ and 2 , respectively; $Z_j \sim U[-\sqrt{3}, \sqrt{3}]$ and $\varphi_j(t) = \sqrt{2} \cos(j\pi t)$. Similarly, the coefficient function $\beta(t)$ is generated according to

$$\beta(t) = \sum_{j=1}^{50} b_j \varphi_j(t),$$

where $b_1 = 0.3$ and $b_j = 4(-1)^{j+1} j^{-2}, j \geq 2$.

- Two cases for the eigenvalues:
 - Case 1 $\gamma = 1.1$, eigenvalues are well spaced and slope function can be well estimated
 - Case 2: $\gamma = 2$, where closely spaced eigenvalues can cause a poor performance for estimating $\beta(t)$
- Truncation of KL expansion: 70% of explained variance
- The experiment was repeated 500 times in each setting
- Criteria:
 - mean bias and standard derivation for ρ
 - MSE evaluated at 100 equidistant point for $\beta(t)$

$$\text{MSE} = \frac{1}{100} \sum_{i=1}^{100} (\hat{\beta}(t_i) - \beta(t_i))^2$$

Comparing FLM(2) and SFLM (1)

ρ	n	$\gamma = 1.1$			$\gamma = 2$		
		bias(sd)	MSE ₁ (sd)	MSE ₂ (sd)	bias(sd)	MSE ₁ (sd)	MSE ₂ (sd)
0	300	-0.0051 (0.0495)	0.0203 (0.0072)	0.0203 (0.0072)	-0.0086 (0.0628)	0.1171 (0.0239)	0.1171 (0.0239)
	500	-0.0003 (0.0428)	0.0087 (0.0030)	0.0087 (0.0029)	-0.0020 (0.0459)	0.0691 (0.0109)	0.0691 (0.0109)
	900	-0.0024 (0.0294)	0.0034 (0.0011)	0.0034 (0.0011)	0.0006 (0.0369)	0.0378 (0.0044)	0.0378 (0.0044)
0.5	300	-0.0062 (0.0457)	0.0201 (0.0071)	0.0267 (0.0101)	-0.0068 (0.0524)	0.1173 (0.0226)	0.1198 (0.0227)
	500	-0.0016 (0.0343)	0.0085 (0.0027)	0.0114 (0.0036)	-0.0037 (0.0400)	0.0689 (0.0101)	0.0702 (0.0102)
	900	-0.0034 (0.0241)	0.0034 (0.0010)	0.0047 (0.0013)	-0.0046 (0.0292)	0.0380 (0.0045)	0.0386 (0.0046)
0.8	300	-0.0062 (0.0261)	0.0202 (0.0069)	0.0836 (0.0361)	-0.0094 (0.0330)	0.1173 (0.0228)	0.1504 (0.0332)
	500	-0.0027 (0.0202)	0.0087 (0.0028)	0.0405 (0.0152)	-0.0057 (0.0245)	0.0689 (0.0110)	0.0876 (0.0161)
	900	-0.0024 (0.0149)	0.0033 (0.0011)	0.0190 (0.0058)	-0.0040 (0.0189)	0.0382 (0.0044)	0.0479 (0.0060)

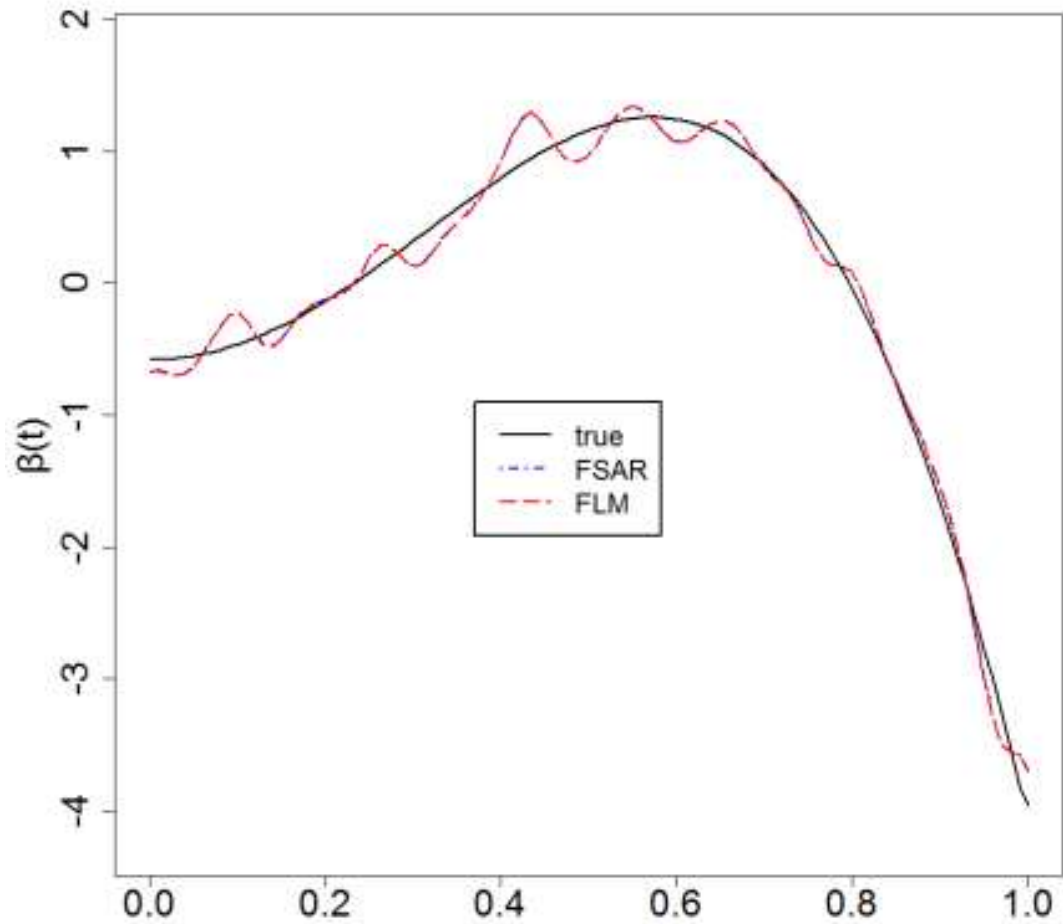
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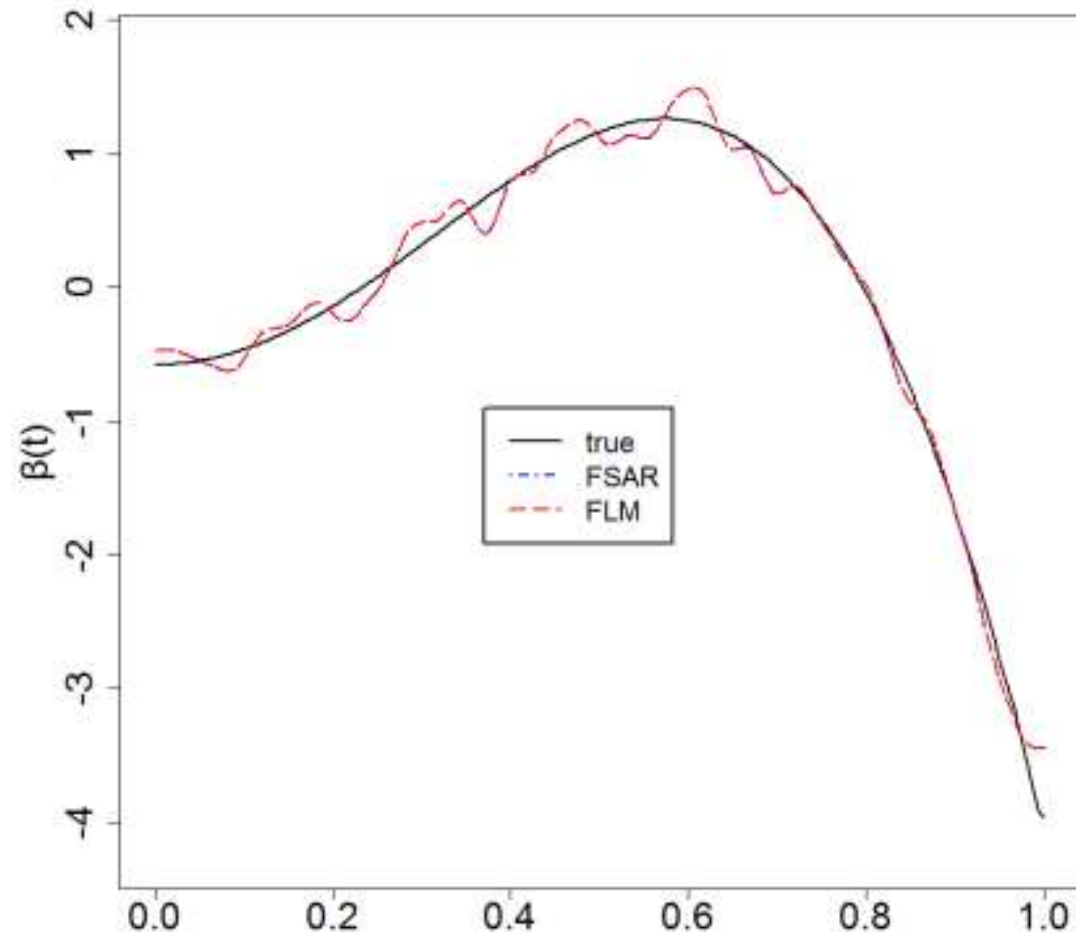
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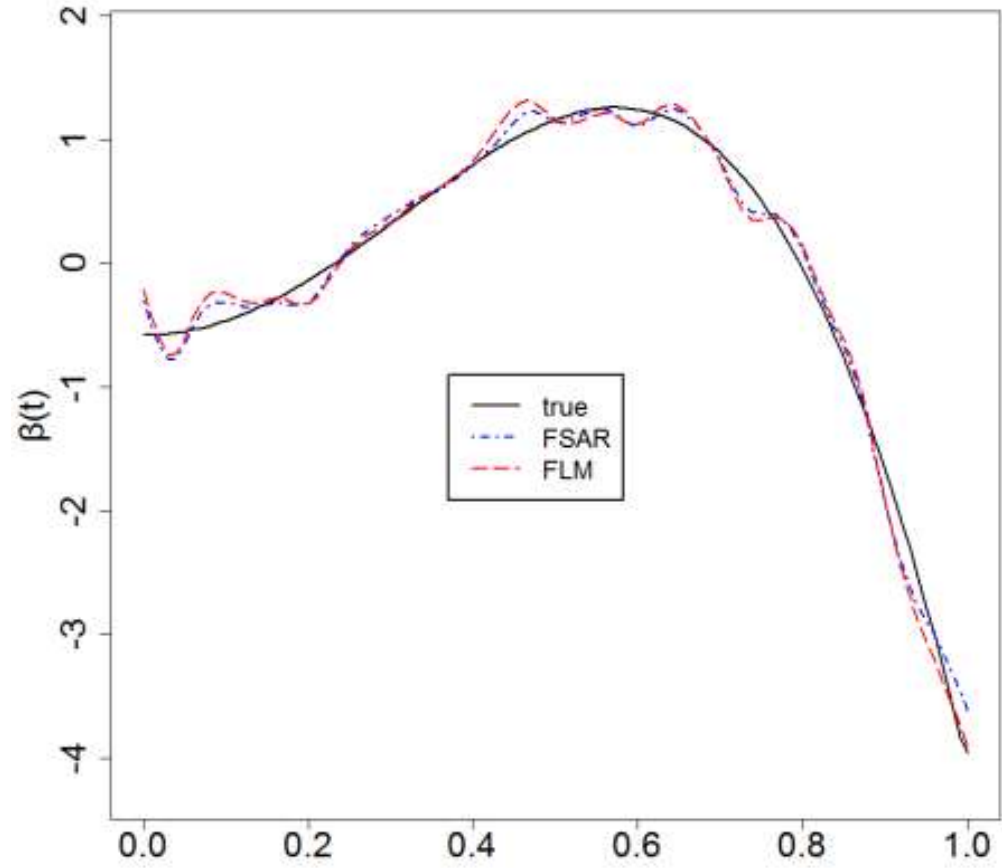
$n=300, \rho=0$



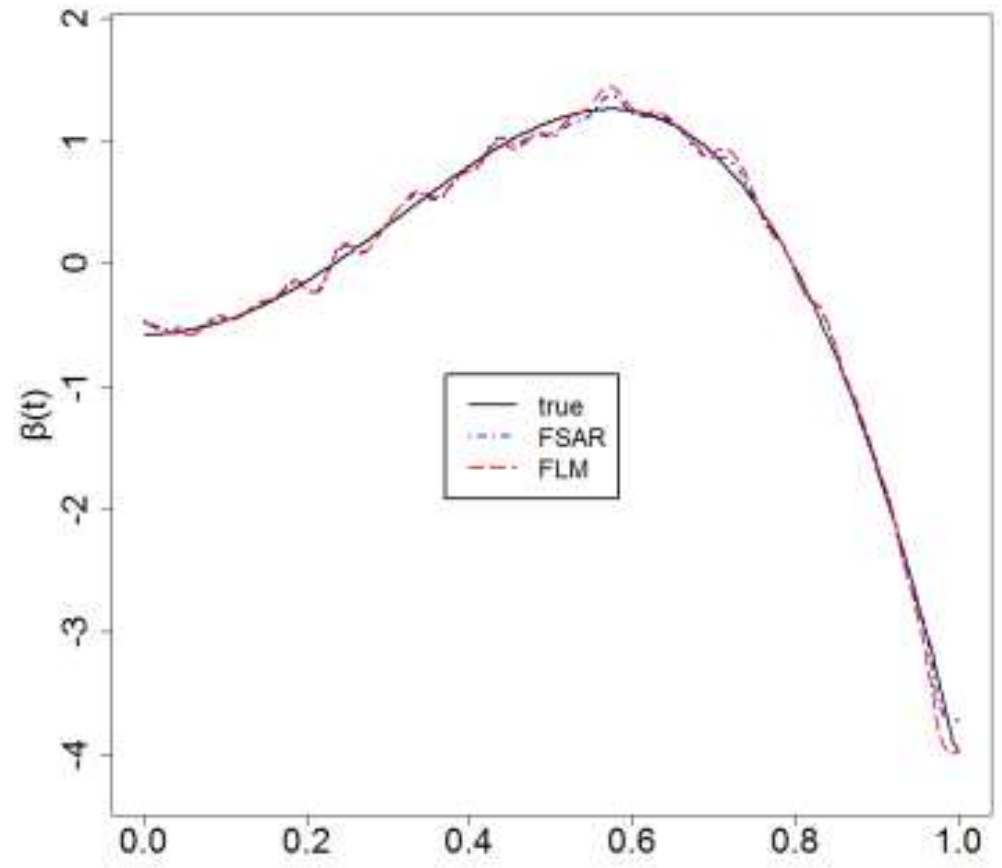
$n=900, \rho=0$



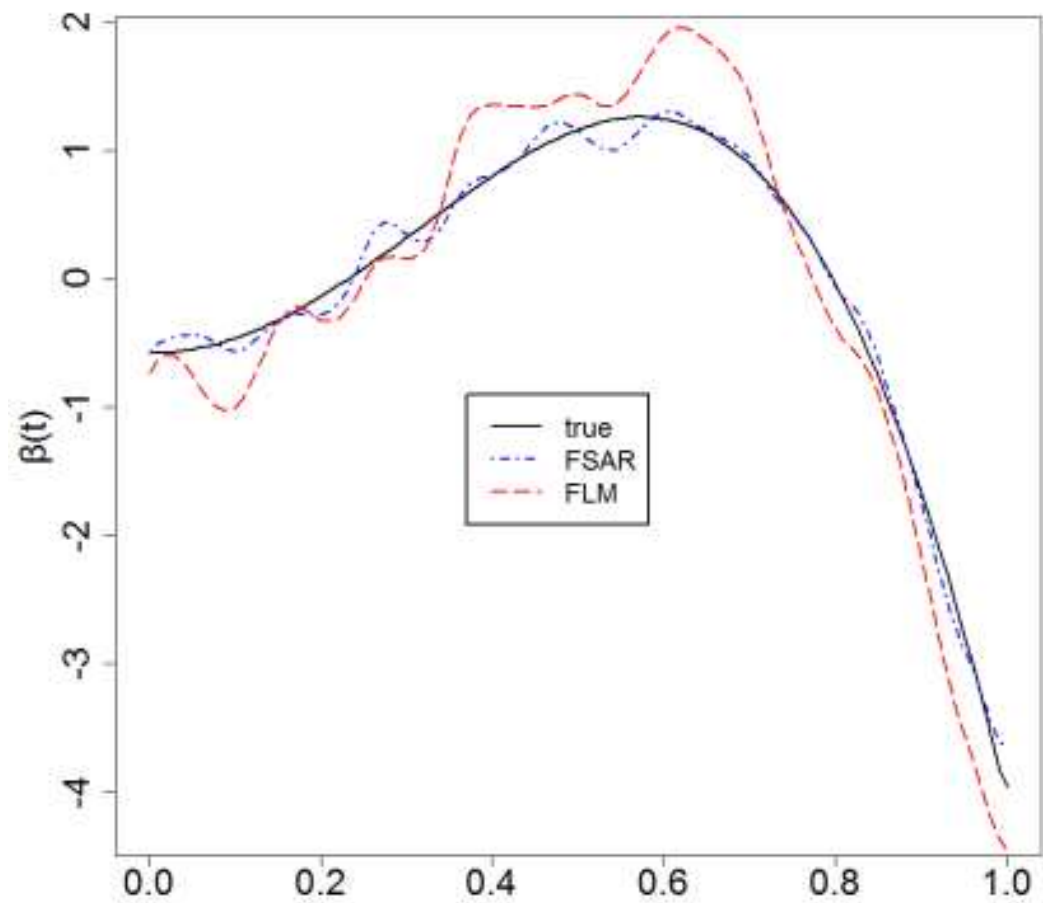
$n=300, \rho=0.3$



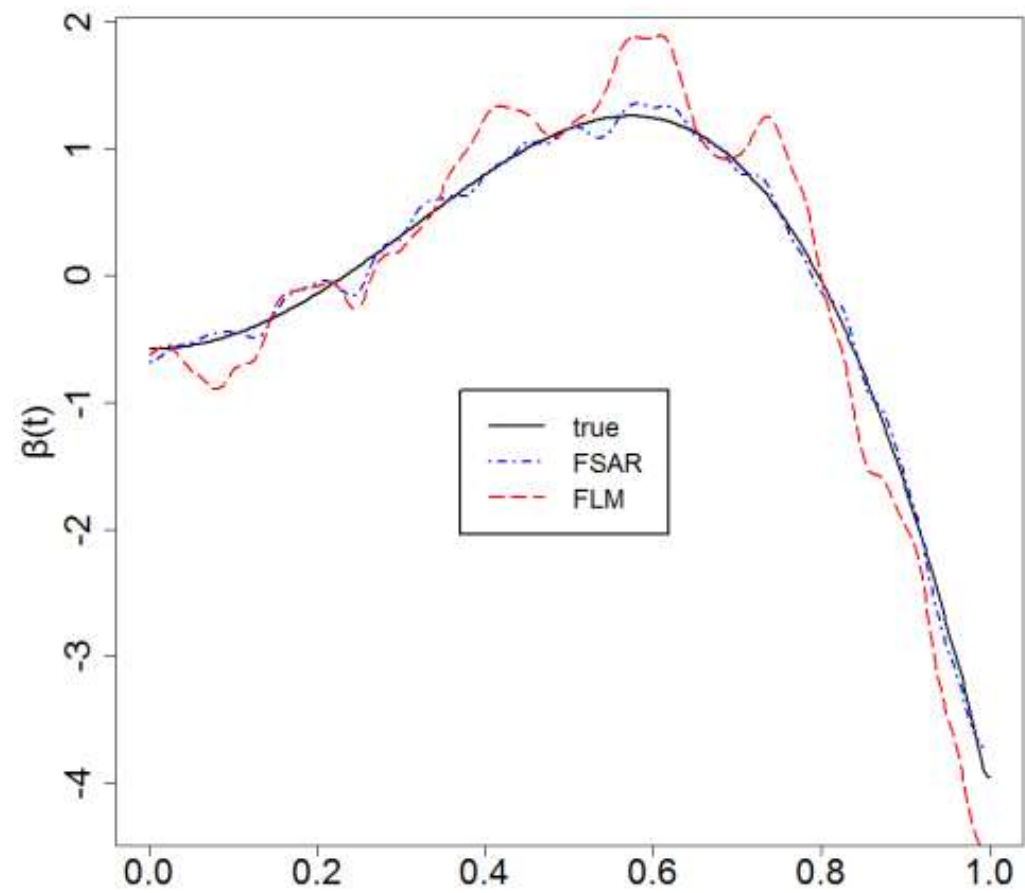
$n=900, \rho=0.3$



$n=300, \rho=0.8$



$n=900, \rho=0.8$



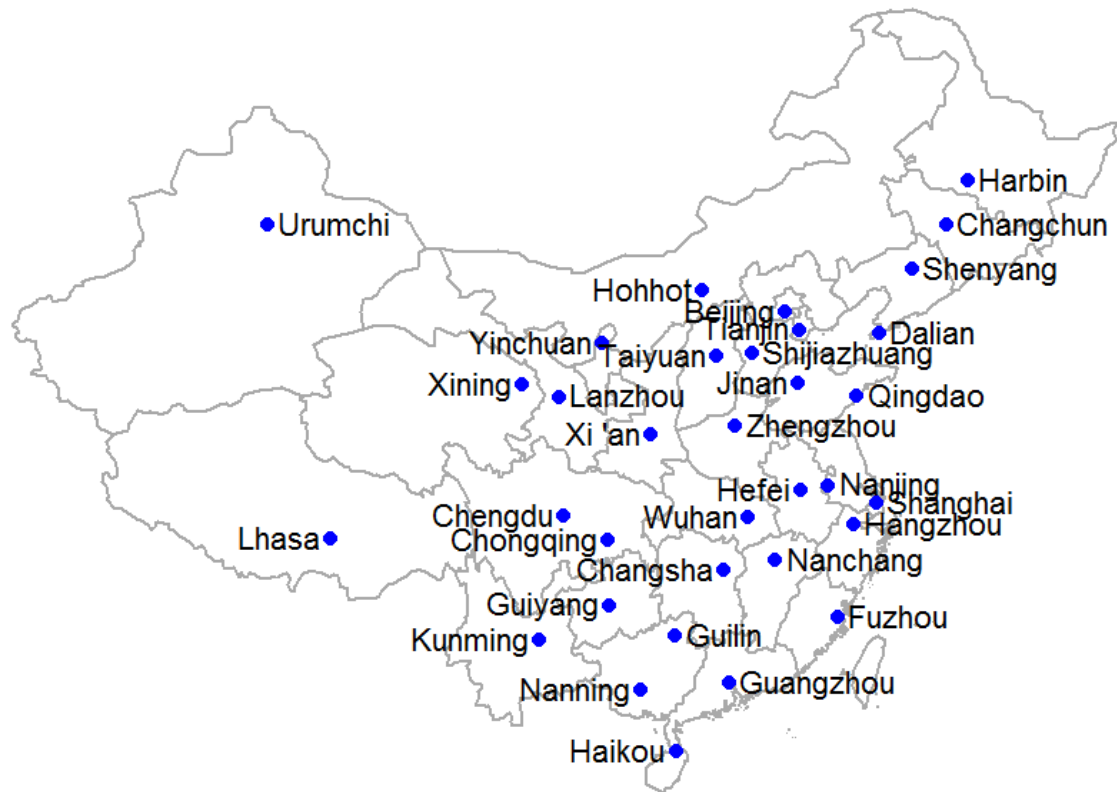
1) When $\rho = 0$, $\hat{\rho}$ is very small, almost equal to zero. And the performances of estimators for $\beta(t)$ based on the proposed method and FPCA based method for FLM perform equally well. This satisfies our expectation, as SFLM reduces to the classical FLM when $\rho = 0$.

2) When $\rho \neq 0$, our proposed method behaves better than FPCA based method, which is as we had expected. MSE of SFLM is always smaller than MSE of FLM when other settings are identical. And as ρ increases, the difference of MSE between SFLM and FLM becomes greater.

3) No matter what ρ equals to, MSE of $\beta(t)$ obtained via SFLM decreases as sample size increases. And the standard deviation of $\hat{\rho}$ has a decreasing pattern, as well. What's more, the bias of $\hat{\rho}$ is small under all cases. Similar to numerical results in Lee (2004), the bias of ρ is negative in all settings.

4) As the case in Hall and Horowitz (2007), $\beta(t)$ is better estimated with $\gamma = 1.1$ than that under $\gamma = 2$ when other simulation parameters equal. The performance of estimator $\hat{\rho}$ is also influenced by α . In case $\gamma = 1.1$, the standard deviation of $\hat{\rho}$ is smaller compared to $\gamma = 2$.

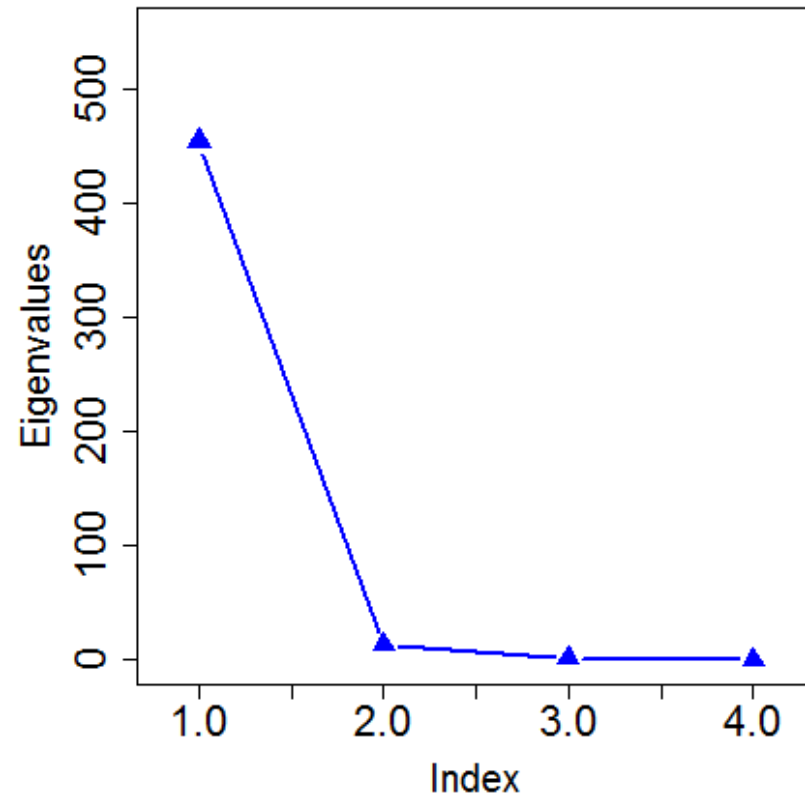
5. Application to chinese weather data



- Investigate the effect of temperature on precipitation
- Y logarithm of mean annual precipitation
- $X(t)$ mean monthly temperature curve
- $n = 34$ cities
- Training set: average over 2005-2007
- Test data: 2008

Main steps

- 1- We smoothed the mean monthly temperature over 3 years by Epanechnikov kernel.
- 2- The spatial weight matrix was formed by nearest k neighbors, each neighbor's weight equalling reciprocal of Euclidean distance $d(i ; i')$ between cities i and i' . For $k = \{2; 3; 4; 5; 6; 7; 8; 9\}$
- 3- Urumchi being too far from other cities was removed
- 4- We perform the SFLM and FLM models to get estimators.
- 5- Then we apply the models to temperature observations in 2008 to predict annual precipitation.



m=2 principal components for FLM and SFLM

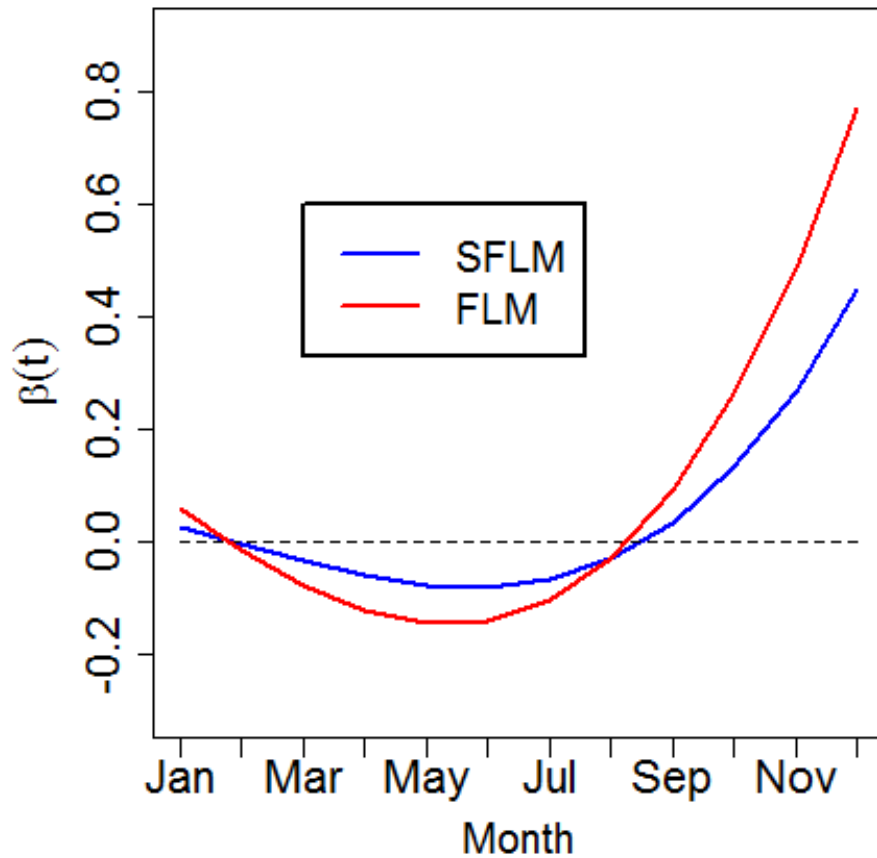
Spatial weight matrix W

	Beijing	Tianjin	Shijiazhuang	Taiyuan	Hohhot	Shenyang	Dalian	Changchun	Harbin	Shanghai	Nanjing	Hangzhou	Hefei	Fuzhou	Nanchang	Jinan	Qingdao	Zhengzhou	Wuhan	Chengde	Guangzhou	Nanning	Gulin	Chengdu	Chongqing	Guiyang	Kunming	Lhasa	Xi'an	Lanzhou	Xining	Yinchuan	Haikou												
Beijing	0	0,42887	0,188946	0,11504	0,105566	0	0	0	0	0	0	0	0	0	0	0,150581	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0											
Tianjin	0,429944	0	0,169031	0	0	0	0,108475	0	0	0	0	0	0	0	0	0,126368	0,109182	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0											
Shijiazhuang	0,202297	0,186197	0	0,272325	0	0	0	0	0	0	0	0	0	0	0	0,181818	0	0,157363	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0										
Taiyuan	0,145568	0	0,321848	0	0,206232	0	0	0	0	0	0	0	0	0	0	0,134192	0	0,19216	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0										
Hohhot	0,186322	0,15436	0,224715	0,287351	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0,147451	0	0											
Shenyang	0	0,127588	0	0	0	0	0,253886	0,31386	0,171255	0	0	0	0	0	0	0	0	0,133411	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0										
Dalian	0,15348	0,18428	0	0	0	0	0,238289	0	0	0	0	0	0	0	0	0	0,163679	0,262271	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0										
Changchun	0	0,090984	0	0	0	0	0,209541	0,129071	0	0,167383	0	0	0	0	0	0	0,093022	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0										
Harbin	0	0,094628	0	0	0	0	0,214923	0,128501	0,467492	0	0	0	0	0	0	0	0,094457	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0										
Shanghai	0	0	0	0	0	0	0	0	0	0	0,225206	0,167469	0,147217	0,112785	0	0	0,094457	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0									
Nanjing	0	0	0	0	0	0	0	0	0	0,188972	0	0,236048	0,113724	0,118379	0	0,122877	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0									
Hangzhou	0	0	0	0	0	0	0	0	0	0,238576	0,245811	0	0,16653	0,128096	0,120386	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0									
Hefei	0	0	0	0	0	0	0	0	0	0,131546	0,155377	0,170292	0	0	0,164813	0	0	0	0,177972	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0									
Fuzhou	0	0	0	0	0	0	0	0	0	0,183462	0,170984	0,239575	0,168114	0	0,237866	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0									
Nanchang	0	0	0	0	0	0	0	0	0	0	0,149794	0	0,19583	0,155255	0	0	0	0	0,272315	0,226816	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0									
Jinan	0,18741	0,255991	0,211353	0	0	0	0	0	0	0	0	0	0	0	0	0	0,188352	0,156894	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0										
Qingdao	0	0,177312	0	0	0	0	0	0,25072	0	0,156887	0,18064	0	0	0	0	0	0,234641	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0										
Zhengzhou	0	0	0,225371	0,23286	0	0	0	0	0	0	0	0	0,165742	0	0	0,293299	0	0	0,182728	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0									
Wuhan	0	0	0	0	0	0	0	0	0	0	0,137895	0	0,205251	0	0,264312	0	0	0,15452	0	0,238223	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0									
Chengde	0	0	0	0	0	0	0	0	0	0	0,254255	0	0,134306	0	0	0	0,275129	0	0,147012	0	0,189299	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0									
Guangzhou	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0,152533	0	0	0	0	0,182987	0	0,190513	0,254879	0	0	0	0	0	0	0	0	0	0	0,218988	0										
Nanning	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0,166873	0	0,262464	0	0	0,194721	0,134711	0	0	0	0	0	0	0,241231	0	0										
Gulin	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0,195041	0,211065	0,248038	0	0	0,198012	0	0	0	0	0	0	0	0,147844	0	0										
Chengdu	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0,335087	0,191209	0,157387	0	0,148687	0,16762	0	0	0	0	0	0										
Chongqing	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0,141406	0,296668	0	0,275923	0,135539	0	0,150464	0	0	0	0	0	0										
Guiyang	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0,185225	0,19931	0,164077	0,267432	0	0,183955	0	0	0	0	0	0	0									
Kunming	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0,188094	0,150802	0,196253	0,192183	0,270021	0	0	0	0	0	0	0										
Lhasa	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0,205551	0	0,214212	0	0	0,187554	0,208725	0	0									
Xi'an	0	0	0	0,200983	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0,215759	0	0	0	0	0	0	0	0	0	0,189942	0	0	0	0	0,188263	0,204953	0									
Lanzhou	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0,144828	0,111308	0	0	0	0,145619	0,168236	0,230006	0								
Xining	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0,144382	0,108325	0	0	0	0,12285	0,433345	0	0,193098	0							
Yinchuan	0	0	0	0,153488	0,163423	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0,196773	0,285468	0,201667	0	0							
Haikou	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0,120115	0,241901	0,30422	0,197292	0	0	0,136472	0	0	0	0

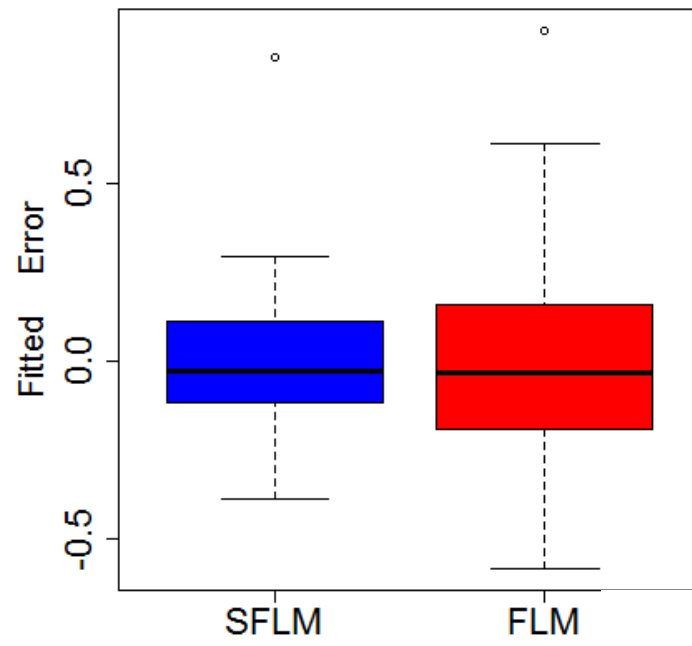
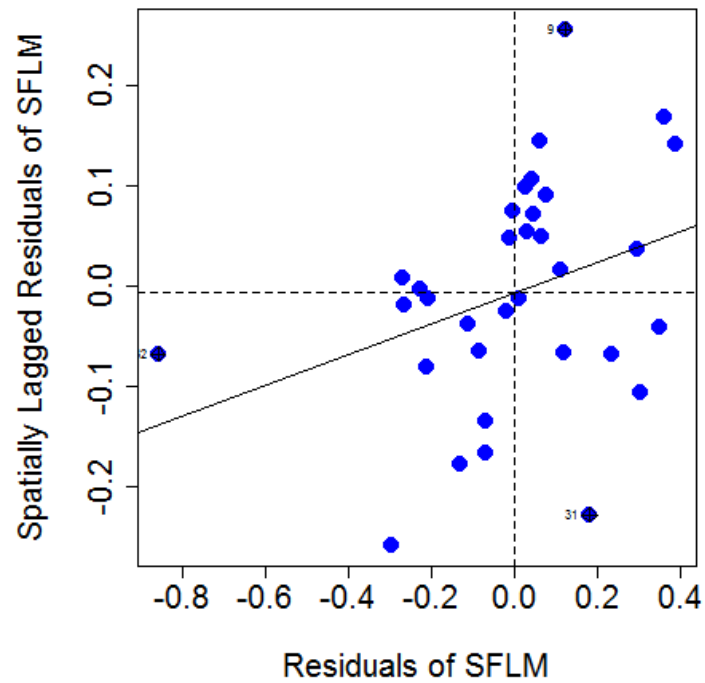
- The optimal number of neighbors for SFLM is $k=5$ according to the procedure of Lesage and Parent (2007)

		$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$	$k = 9$
$m = 2$	L	-4.42	-3.28	-1.51	-1.11	-1.90	-2.15	-2.25	-2.50
	B	90.39	302.32	1755.32	2253.14	1163.4	943.31	887.44	695.95

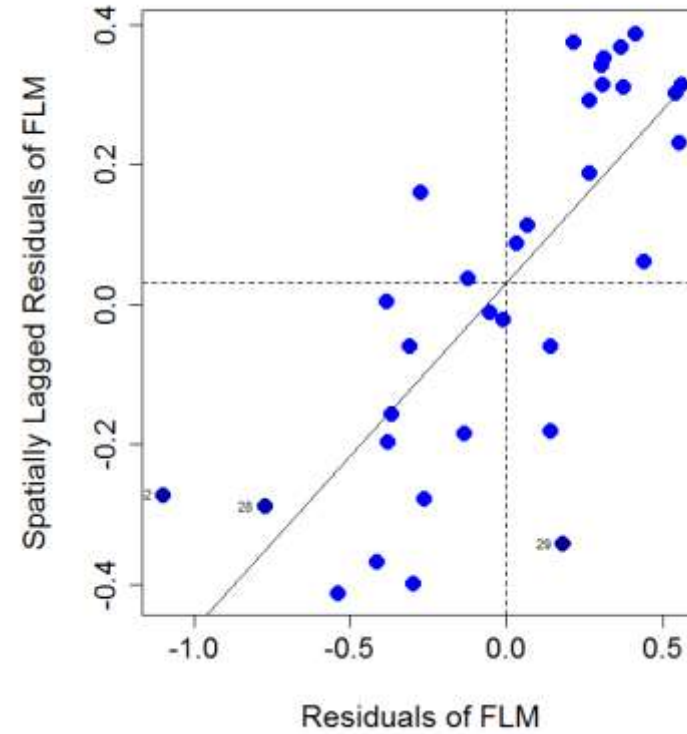
- B a bayesian model probability ratio
- $\hat{\rho} = 0.58$



- The two estimated curves $\beta(t)$ have similar shapes
- Precipitation is much more influenced by temperature in winter than in other seasons. Under SFLM the precipitation of each city is less affected by temperature during the whole year.



Moran I = 0.15



Moran I = 0.41

- SFLM is more precise than FLM
- A majority of spatial autocorrelation in responses have been removed.

6. Conclusions and future works

- SFMLM is simple and efficient when spatial autocorrelation is present in the response
- Use of PLS components instead of PCA (cf Preda and Saporta, 2005)
- Spatial cross validation
- Extensions to multiple functional predictors and spatially dependent predictors

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