Optimal interval observer for switched Takagi-Sugeno systems: an application to interval fault estimation

Yosr Garbouj, Thach Ngoc Dinh, Tarek Raïssi, Talel Zouari, Moufida Ksouri

Optimal interval observer for switched Takagi-Sugeno systems: an application to interval fault estimation

Yosr Garbouj#, Thach N. Dinh#,⋆, Tarek Raïssi, Talel Zouari and Moufida Ksouri

***Abstract*—The main goal of this paper is to design interval observers for continuous nonlinear switched systems. The nonlinear modes are described by the multimodel approach of Takagi-Sugeno (T-S) fuzzy systems where premise variables depending on the state vector which is unmeasurable. In this paper, we propose T-S interval observers that consider the unmeasurable premise variables as bounded uncertainties under common assumptions that additive disturbances as well as measurement noises are unknown but bounded. The stability and the nonnegativity conditions are given in terms of Linear Matrix Inequality (LMI) to ensure simultaneously the convergence and the nonnegativity of error dynamics. Furthermore, in the absence of measurement noises, optimal gains attenuating the effect of additive disturbances are computed using H_∞ approach to improve the accuracy of the present interval observers. Theoretical results are finally applied to a numerical example to highlight the performance of the proposed method.**

***Index Terms*—Nonlinear switched systems, T-S interval observer, unmeasurable premise variable, stability, H_∞ formalism.**

I. Introduction

MANY engineering applications evolve through the coupling between continuous and discrete dynamics where a collection of indexed modes interacts with a switching signal that selects the active one at each time instant. Such systems are called *hybrid systems* which have attracted an ever growing attention [1], [2]. Meanwhile, *switched systems* are an important class of hybrid systems [3] and many great efforts have been paid to the studies of this class due to its appearance in a large number of physical applications such as biological systems [4], robotic systems [5], embedded systems [6], etc. Among these studies, some interesting results have been reported to deal with the problems of robust control [7], diagnosis [8] while others are particularly devoted to state estimation of such systems. More concretely, the case of linear switched systems are usually inspected [9]–[12]. However, the case of *nonlinear switched systems* has not been fully studied yet. Although in practice it is well-known that various domains have to be described by nonlinear switched behaviors such as automotive [13], converters [14], network control applications [15] and so on.

Since several decades, *Takagi-Sugeno (T-S) fuzzy systems* have been considered as a powerful tool to cope with nonlinearities [16]. This approach is useful to analyze nonlinear systems by providing an effective representation of them [17]. T-S fuzzy models are based on fuzzy sets applied to a set of linear local models decomposing a nonlinear system into different zones. The validity of each one is quantified by a nonlinear weighting function depending on the so-called premise variables (or decision variables). Two main cases can be considered for premise variables: some works are devoted to the premise variables which are measurable (as the input or the output of the system) [18] while most of the literature deals with the second case for which the premise variables are composed of a subset of the system state $x(t)$ [19], [20]. As widely known the former is conservative. Unlike the former, the latter is very challenging and appears in most of nonlinear mechanical systems. e.g., the readers can for instance refer to [21] for an example of a simple nonlinear mass-spring-damper mechanical system. Since the premise variables are endogenous and unknown, the T-S model defines a class of nonlinear systems with unknown nonlinearities which often hamper and sometimes prohibit construction of observers.

Following the remarkable development of fuzzy control systems, nonlinear switched plants can be also described in the compact form of T-S fuzzy models. In the literature, few works are devoted to this family of systems [18], [22], [23]. However, the previously mentioned works adopt a restrictive requirement of the premise variables: *they are supposed to be measurable*.

Recently, designing observers for such switching representations with unmeasureable premise variables becomes an open issue to tackle. In addition, the case of systems subject to disturbances and measurement noises is more challenging as it is the case for most of real-life systems. However, this case has not been fully investigated in the literature. Conventional observer may not be an efficient method to deal with uncertainties. Additionally, for the purpose of control such as stabilization and tracking, precise information of the state vector in transient periods is not necessary. However, practically there is a great demand for estimation of the state of a system with guarantees at all time and the notion of *interval observer* has been one of useful approaches to meeting this practical demand. That is why interval observers have

# These authors contributed equally to this work.

⋆ Correspondence: Thach N. Dinh (e-mail: ngoc-thach.dinh@lecnam.net).
Thach Ngoc Dinh and Tarek Raïssi are with the Conservatoire National des Arts et Métiers (CNAM), Cedric-LAETITIA, 292 rue St-Martin, 75141 Paris Cedex 03, France.
Yosr Garbouj, Talel Zouari and Moufida Ksouri are with the University of Tunis El Manar, National Engineering School of Tunis (ENIT), Analysis, Conception and Control of Systems Laboratory (LR-11-ES20), BP 37, Le Belvedere 1002, Tunis, Tunisia.
Talel Zouari is also with the Department of Electromechanical Engineering, ESPRIT School of Engineering, Tunis, Tunisia.

been fertile ground for studies in the past few years. In the literature, interval observers have been successfully applied to many physical systems such as biotechnological processes [24], biochemical processes [25], energy generation system [26]. It has been shown in many contributions that interval observers do not only give interval estimates but also can be employed for stabilization by feedback control [27]–[29]. Although design of interval observers requires bounds of the uncertainties/disturbances and bounds of the initial conditions to be known a priori, this requirement is accepted in many applications. Interval observer designs for several classes of systems have been reviewed in both continuous and discrete time, e.g., linear systems [30]–[33], bilinear systems [28], nonlinear systems [34]–[36], time-delay systems [37], linear switched systems [38]–[41].

According to the above-mentioned studies, few results of interval observer designs are devoted to nonlinear switched systems represented by T-S fuzzy models. It is worth noting that most of existing works in the literature concerning interval estimator design for fuzzy systems, even in the non-switching case, have coped only with the *measurable* premise variables [26], [42]. To the best of the authors' knowledge, designing interval observers with the constraints of *unmeasurable* premise variables has not been fully considered yet. This motivates the present work. The goal of this paper is to propose new results of T-S interval observer designs for nonlinear switched systems subject to measurement noises and additive disturbances. Using the center-of-gravity method for defuzzification, nonlinear switched systems can be represented by T-S fuzzy models with unmeasurable premise variables. The first contribution is to construct interval observers for this class of fuzzy systems. The stability and the nonnegativity properties are both given in terms of Linear Matrix Inequalities (LMIs). The former is achieved via a common Lyapunov function while the latter is inspired by a simple but interesting fact introduced in [43] (see Lemma 3). The second contribution of this paper is to deal with the problem of the interval width optimization in the absence of measurement noises. To the best of the authors'knowledge, this optimization issue has not been fully treated. Only the case of discrete-time linear systems has been considered in [44]. The idea is based on H_∞ *approach* to compute optimal gains which attenuate the effect of the system's disturbances on the estimation error. Thus, the optimal gains ensure a tighter interval width which improves the accuracy of the interval between the lower and upper bounds.

Compared to previous works, the contributions of this paper are summarized as follows:

1) The proposed scheme is applicable to T-S model with premise variables composed of a subset of the system state. Note that most of the designs available in the literature are carried out for cooperative systems (sometimes after a coordinate transformation) but the key advantage of the new construction is the simplicity of its dynamics. Each copy of observer, or its associated error equation, does not possess the property of being a cooperative or a nonnegative system. It is the first time such design is introduced.
2) For a given lower and upper bounds of disturbances and in the absence of measurement noises, an optimal interval estimation for the state can be obtained based on an energy-bounded design method. To the best of our knowledge, this is the first interval observer that can attenuate the effect of disturbances for T-S systems with unmeasurable premise variables.
3) An application to sensor fault detection is proposed. Simulations are given to highlight the effectiveness of the proposed schemes.

The paper has the following structure: preliminaries and the system description are given in Section II. Section III is devoted to the main results: (i) designing T-S interval observers, (ii) based on H_∞ *approach*, optimizing the interval width by computing optimal gains and (iii) a fault detection scheme is introduced as an application. Two examples are considered in Section IV: the first one draws comparative simulations while the second one illustrates the efficiency of the proposed method in the sensor fault detection. Section V concludes the paper.

II. Preliminaries and system description

A. Notions, definitions and lemmas

The set of real numbers is denoted by $\Re$ and the set of nonnegative real numbers is denoted by $\Re_{\geq 0}$, i.e., $\Re_{\geq 0} := [0, +\infty)$ where $\Re^{n\times n}_{\geq 0}$ is of dimension $n \times n$. Inequalities are understood *component-wise*, i.e., for $x_a = [x_{a,1}, ..., x_{a,n}]^\top \in \Re^n$ and $x_b = [x_{b,1}, ..., x_{b,n}]^\top \in \Re^n$, $x_a \leq x_b$ if and only if, for all $i \in \{1, ..., n\}$, $x_{a,i} \leq x_{b,i}$. I, 0 denote respectively identity and zero matrices with appropriate dimensions. A function $\alpha : \Re_{\geq 0} \to \Re_{\geq 0}$ is said to be positive definite and written as $\alpha \in \mathcal{P}$ if it is continuous and satisfies $\alpha(0) = 0$ and $\alpha(s) > 0$ for all $s \in (0, \infty)$. A function $\alpha \in \mathcal{P}$ is said to be of class $\mathcal{K}$ if it is strictly increasing. If in addition α is unbounded, then it is of class $\mathcal{K}_\infty$. The symbol $P \succ 0$ (resp. $P \prec 0$) means that the symmetric matrix P is positive (resp. negative) definite. E_p is a $(p \times 1)$ vector whose elements are equal to 1. I_n is the identity matrix with dimension $n \times n$. For $x(t) : \Re_{\geq 0} \to \Re^n$, the $\mathcal{L}_2$ norm is denoted by $\|x(t)\|_2$. For a measurable and locally essentially bounded input $u : \Re_{\geq 0} \to \Re$, the symbol $\|u\|_{[t_0,t_1]}$ denotes its $\mathcal{L}_\infty$ norm. If $t_1 = +\infty$, then we will simply write $\|u\|$. The $\mathcal{L}_\infty$ is denoted as the set of all inputs u such that $\|u\| < \infty$. The left and right endpoints of an interval $[x(t)]$ are denoted by $\underline{x}(t)$ and $\overline{x}(t)$ such as $[x(t)] = [\underline{x}(t), \overline{x}(t)]$. A matrix $A \in \Re^{n\times n}$ is called Metzler if all the off-diagonal elements are nonnegative. A matrix $A \in \Re^{n\times n}$ is said to be nonnegative if each entry of A is nonnegative. Given a matrix $A \in \Re^{m\times n}$, we define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ and denote the absolute value of a matrix by $|A| = A^+ + A^-$ (similarly for vectors). For square matrices T_i, we define $diag([T_1 \dots T_N]) = \begin{bmatrix} T_1 & 0 & \cdots & 0 \\ 0 & T_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & T_N \end{bmatrix}$.

Lemma 1: [45] The system described by:

$$\dot{x}(t) = Ax(t) + u(t) \tag{1}$$

is said to be nonnegative if A is a Metzler matrix and $u(t) \geq 0$. For any initial condition $x(0) \geq 0$, the solution of (1) satisfies $x(t) \geq 0, \forall t \geq 0$.

Lemma 2: [46] Let $x \in \Re^n$ be a vector such that $\underline{x} \leq x \leq \overline{x}$. (1) if $A \in \Re^{m\times n}$ is a constant matrix, then

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x} \tag{2}$$

(2) if $A \in \Re^{m\times n}$ is a matrix satisfying $\underline{A} \leq A \leq \overline{A}$, for some $\underline{A}, \overline{A} \in \Re^{m\times n}$, then

$$\begin{aligned} &\underline{A}^+\underline{x}^+ - \overline{A}^+\underline{x}^- - \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^- \leq Ax \\ &\leq \overline{A}^+\overline{x}^+ - \underline{A}^+\overline{x}^- - \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^- \end{aligned} \tag{3}$$

Lemma 3: [43] A matrix $A \in \Re^{n\times n}$ is Metzler if and only if there exists $\eta \in \Re_{\geq 0}$ such that $A + \eta I \in \Re^{n\times n}_{\geq 0}$.

Consequently, if there exist a positive diagonal matrix $P \in \Re^{n\times n}$ and a constant $\eta > 0$ such that

$$PA + \eta P \geq 0, \tag{4}$$

then, A is Metzler.

Lemma 4: [47] Consider x and y with appropriate dimensions and Ω a positive definite matrix. the following property is verified:

$$x^Ty + y^Tx \leq x^T\Omega x + y^T\Omega^{-1}y \tag{5}$$

Lemma 5: [48] Let $\lambda > 0$ be a scalar and $P \in \Re^{n\times n}$ be a symmetric positive definite matrix, then:

$$2x^Ty \leq \frac{1}{\lambda}x^TPx + \lambda y^TP^{-1}y \qquad x, y \in \Re^n \tag{6}$$

Lemma 6 (Input-to-State Stability (ISS) [49]): Consider the linear switched system with inputs

$$\dot{x}(t) = A^\sigma x(t) + d(t),\ \sigma \in \{1, 2, \ldots, N\} \tag{7}$$

Suppose that there exist a continuously differentiable positive definite, radially unbounded function $V : \Re^n \to [0, \infty)$, a class $\mathcal{K}_\infty$ function α, a class $\mathcal{K}$ function γ and a number $\wp > 0$ such that

$$\dot{V} \leq -\wp\alpha(V) + \gamma(\|d\|) \tag{8}$$

along all trajectories of (7). Then, the linear switched system (7) is Input-to-State Stable (ISS). The ISS is used to study the stability of a system with external inputs. The system is ISS if it is globally asymptotically stable in the absence of external inputs and if its trajectories are bounded by a function of the size of the input for all sufficiently large times.

Definition 1 (Interval observer for switched systems [50]): Consider a switched system:

$$\begin{cases} x(k+1) = f_q(x(k), d(k)), \\ y(k) = g_q(x(k)), \end{cases} \tag{9}$$

with the state $x \in \mathbb{R}^n$, the output $y \in \mathbb{R}^p$, the index of the active subsystem $q \in \overline{1, N}$, the number of subsystems is $N \in \mathbb{N}$ and f_q, g_q are functions. The uncertainties $d(k) \in \mathbb{R}^\ell$ are such that there exists a sequence $\overline{d}(k) \in \mathbb{R}^\ell$ where, for all $k \geq 0$, $-\overline{d}(k) \leq d(k) \leq \overline{d}(k)$. The initial condition $x(0)$, is assumed to be bounded by two known bounds:

$$\underline{x}(0) \leq x(0) \leq \overline{x}(0). \tag{10}$$

Then, the dynamical system

$$z(k+1) = h_q\left(z(k), y(k), \overline{d}(k)\right), q \in \overline{1, N},\ N \in \mathbb{N}, \tag{11}$$

associated with the initial condition $z(0) = r_q(\overline{x}(0), \underline{x}(0)) \in \mathbb{R}^{n_z}$ and bounds for the solution $x(k)$: $\overline{x}(k) = \overline{h}_q(z(k), y(k))$, $\underline{x}(k) = \underline{h}_q(z(k), y(k))$, where $q \in \overline{1, N},\ N \in \mathbb{N}$, h_q, r_q, $\overline{h}_q$ and $\underline{h}_q$ are functions, is called (i) a framer for (9) if for any vectors $x(0), \underline{x}(0)$ and $\overline{x}(0)$ in $\mathbb{R}^n$ satisfying (10), the solutions denoted respectively x and z of (9)-(11) with respectively $x(0)$, $z(0) = r_q(\overline{x}(0), \underline{x}(0))$ as initial condition at 0, satisfy for all $k \geq 0$, the inequalities

$$\underline{x}(k) = \underline{h}_q(z(k), y(k)) \leq x(k) \leq \overline{h}_q(z(k), y(k)) = \overline{x}(k), \tag{12}$$

(ii) an interval observer for (9) if in addition $|\overline{h}_q(z(k), y(k)) - \underline{h}_q(z(k), y(k))|$ is input-to-state stable (ISS) with respect to $d(k) \in \Re^\ell$ for all $q \in \overline{1, N},\ N \in \mathbb{N}$.

B. Uncertain T-S model formulation for nonlinear switched systems

Consider a continuous time nonlinear switched system described as follows :

$$\begin{aligned} &\Sigma_{\sigma(t)} : \begin{cases} \dot{x}(t) = f_{\sigma(t)}\left(x(t), u(t), d(t)\right) \\ y(t) = g_{\sigma(t)}\left(x(t), v(t)\right) \end{cases} \\ &\sigma(t) : \Re_{\geq 0} \to \{1, 2, \ldots, N\} \end{aligned} \tag{13}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the input and $y(t) \in \Re^p$ is the output. $d(t) \in \Re^n$ and $v(t) \in \Re^p$ are respectively the bounded additive disturbances and measurement noises. $\sigma(t)$ is the switching law such that $\sigma(t) \in \{1, ..., N\}$ is the index of the active mode. For example, if one has $\sigma(t) = i,\ i \in \{1, 2, \cdots, N\}$, the system is said to be in the mode i at the instant t. For the sake of simplicity, $\sigma(t)$ will be simply replaced by σ. f_σ and g_σ are nonlinear functions. The additive disturbances $d(t)$ and the measurement noises $v(t)$ are assumed to be unknown but bounded.

The first step is to approximate each nonlinear mode σ of the system (13) by T-S fuzzy models which consist of a set of linear sub-models interpolated through a weighting function $\mu_i^\sigma(\xi(t))$ to contribute to the global behavior of the nonlinear switched system. Each linear model represents a local dynamics of the whole system. By using the center-of-gravity method [51], for defuzzification of the possible sets, the T-S representation is given by the following compact form:

$$\begin{cases} \dot{x}(t) = \sum\limits_{i=1}^{r} \mu_i^\sigma(\xi(t))(A_i^\sigma x(t) + B_i^\sigma u(t)) + d(t) \\ y(t) = C^\sigma x(t) + v(t) \\ \forall\sigma \in \{1, 2, \ldots, N\}, \forall i \in \{1, \ldots, r\}, \end{cases} \tag{14}$$

where $A_i^\sigma \in \Re^{n\times n}, B_i^\sigma \in \Re^{n\times m}$ *and* $C^\sigma \in \Re^{p\times n}$ are known constant matrices. $\mu_i^\sigma(\xi(t))$ are the weighting functions depending on the so-called decision variable $\xi(t)$ that can be internal or external to the system. When these variables are internal, they can be measurable such as the input or the output of the system (i.e. $\{u(t), y(t)\}$) or unmeasurable as the state of the system (ie. $x(t)$). The weighting functions verify the following convex sum properties:

$$\left\{\begin{array}{l} 0 \leq \mu_i^\sigma(\xi(t)) \leq 1,\ \forall\sigma \in \{1, ..., N\}, \forall i \in \{1, \dots r\} \\ \sum\limits_{i=1}^{r} \mu_i^\sigma(\xi(t)) = 1 \end{array}\right. \tag{15}$$

Assumption 1:

$$\underline{d} \leq d(t) \leq \overline{d},\ |v(t)| \leq \overline{V} E_p,\ \forall t \geq 0 \tag{16}$$

where $\underline{d} = -\overline{d} \in \Re^n$ and the scalar $\overline{V}$ are a priori known. Assumption 1 is basic in the literature of interval observers where the uncertainties are assumed bounded with known bounds.

Assumption 2: The state of the system $x(t)$ and the known input vector $u(t)$ are supposed to be bounded in norm.

In this work, we suppose that the weighting functions depend on the system state which is unmeasurable (i.e. $\xi(t) = x(t)$). It is worth pointing out that considering the unknown state as a decision variable will improve the synchronization process of control or diagnosis of the system which is an important advantage of this system's structure. Therefore, the system (14) can be rewritten as:

$$\left\{\begin{array}{l} \left\{\begin{array}{l} \dot{x}(t) = \sum\limits_{i=1}^{r} \mu_i^\sigma(x(t))(A_i^\sigma x(t) + B_i^\sigma u(t)) + d(t) \\ y(t) = C^\sigma x(t) + v(t) \end{array}\right. , \\ \forall\sigma \in \{1, 2, \dots, N\}, \forall i \in \{1, \dots, r\}. \end{array}\right. \tag{17}$$

Given the lower and upper bounds $\underline{x}(t), \overline{x}(t), \in \Re^n$ of the state $x(t)$. Then, inspired by the work [52], by adding and subtracting at the same time, respectively, the $\sum\limits_{i=1}^{r} \mu_i^\sigma(\overline{x}(t))(A_i^\sigma x(t) + B_i^\sigma u(t))$ term or the $\sum\limits_{i=1}^{r} \mu_i^\sigma(\underline{x}(t))(A_i^\sigma x(t) + B_i^\sigma u(t))$ term and after rearranging, the second step consists in rewriting the system (17) with unmeasurable premise variables to an uncertain system with upper and lower estimated premise variables. The state vector is given in two equivalent ways as follows $\forall\sigma \in \{1, 2, \dots, N\}$ and $\forall i \in \{1, \dots, r\}$:

$$\left\{\begin{array}{l} \quad \dot{x}(t) = \sum\limits_{i=1}^{r} \mu_i^\sigma(\overline{x}(t))(A_i^\sigma x(t) + B_i^\sigma u(t)) \\ \qquad - \sum\limits_{i=1}^{r} \overline{\delta}_i^\sigma(t)(A_i^\sigma x(t) + B_i^\sigma u(t)) + d(t) \\ \text{or} \\ \quad \dot{x}(t) = \sum\limits_{i=1}^{r} \mu_i^\sigma(\underline{x}(t))(A_i^\sigma x(t) + B_i^\sigma u(t)) \\ \qquad + \sum\limits_{i=1}^{r} \underline{\delta}_i^\sigma(t)(A_i^\sigma x(t) + B_i^\sigma u(t)) + d(t) \\ y(t) = C^\sigma x(t) + v(t) \end{array}\right. \tag{18}$$

with

$$\begin{array}{l} \overline{\delta}_i^\sigma(t) = \mu_i^\sigma(\overline{x}(t)) - \mu_i^\sigma(x(t)) \\ \underline{\delta}_i^\sigma(t) = \mu_i^\sigma(x(t)) - \mu_i^\sigma(\underline{x}(t)) \end{array}$$

Thus, using $\sum\limits_{i=1}^{r} \mu_i^\sigma(\overline{x}(t)) = \sum\limits_{i=1}^{r} \mu_i^\sigma(\underline{x}(t)) = 1$ for all $\underline{x}(t) \in \Re^n$ and $\overline{x}(t) \in \Re^n$, we obtain:

$$\left\{\begin{array}{l} \quad \dot{x}(t) = \sum\limits_{i=1}^{r} \mu_i^\sigma(\overline{x}(t))[(A_i^\sigma - \overline{\Delta A}^\sigma(t))x(t) + \\ \qquad (B_i^\sigma - \overline{\Delta B}^\sigma(t))u(t)] + d(t) \\ \text{or} \\ \quad \dot{x}(t) = \sum\limits_{i=1}^{r} \mu_i^\sigma(\underline{x}(t))[(A_i^\sigma + \underline{\Delta A}^\sigma(t)x(t) + \\ \qquad (B_i^\sigma + \underline{\Delta B}^\sigma(t)u(t)] + d(t) \\ y(t) = C^\sigma x(t) + v(t) \end{array}\right. \tag{19}$$

where the uncertainties $\overline{\Delta A}^\sigma(t)$, $\overline{\Delta B}^\sigma(t)$ or $\underline{\Delta A}^\sigma(t)$, $\underline{\Delta B}^\sigma(t)$ are given by:

$$\begin{array}{ll} \overline{\Delta A}^\sigma(t) = \sum\limits_{i=1}^{r} \overline{\delta}_i^\sigma(t) A_i^\sigma, & \underline{\Delta A}^\sigma(t) = \sum\limits_{i=1}^{r} \underline{\delta}_i^\sigma(t) A_i^\sigma \\ \quad = \mathcal{A}^\sigma \overline{\Sigma}_A^\sigma(t) \mathrm{E}_A & \quad = \mathcal{A}^\sigma \underline{\Sigma}_A^\sigma(t) \mathrm{E}_A \end{array} \tag{20}$$

$$\mathcal{A}^\sigma = \left[\begin{array}{ccc} A_1^\sigma & \cdots & A_r^\sigma \end{array}\right],\ \mathrm{E}_A = \left[\underbrace{\begin{array}{ccc} I_n & \cdots & I_n \end{array}}_{r \text{ terms}}\right]^T \tag{21}$$

$$\overline{\Sigma}_A^\sigma(t) = diag([\overline{\delta}_1^\sigma(t) I_n \dots \overline{\delta}_r^\sigma(t) I_n]) \tag{22}$$

$$\underline{\Sigma}_A^\sigma(t) = diag([\underline{\delta}_1^\sigma(t) I_n \dots \underline{\delta}_r^\sigma(t) I_n]) \tag{23}$$

$$\begin{array}{ll} \overline{\Delta B}^\sigma(t) = \sum\limits_{i=1}^{r} \overline{\delta}_i^\sigma(t) B_i^\sigma, & \underline{\Delta B}^\sigma(t) = \sum\limits_{i=1}^{r} \underline{\delta}_i^\sigma(t) B_i^\sigma \\ \quad = \mathcal{B}^\sigma \overline{\Sigma}_B^\sigma(t) \mathrm{E}_B & \quad = \mathcal{B}^\sigma \underline{\Sigma}_B^\sigma(t) \mathrm{E}_B \end{array} \tag{24}$$

$$\mathcal{B}^\sigma = \left[\begin{array}{ccc} B_1^\sigma & \cdots & B_r^\sigma \end{array}\right],\ \mathrm{E}_B = \left[\underbrace{\begin{array}{ccc} I_m & \cdots & I_m \end{array}}_{r \text{ terms}}\right]^T \tag{25}$$

$$\overline{\Sigma}_B^\sigma(t) = diag([\overline{\delta}_1^\sigma(t) I_m \dots \overline{\delta}_r^\sigma(t) I_m]) \tag{26}$$

$$\underline{\Sigma}_B^\sigma(t) = diag([\underline{\delta}_1^\sigma(t) I_m \dots \underline{\delta}_r^\sigma(t) I_m]) \tag{27}$$

Remark 1: Due to the convex property of the weighting functions (15), we have $-1 \leq \overline{\delta}_i^\sigma(t) \leq 1$ and $-1 \leq \underline{\delta}_i^\sigma(t) \leq 1$. The terms $\overline{\Sigma}_A^\sigma(t)$, $\underline{\Sigma}_A^\sigma(t)$, $\overline{\Sigma}_B^\sigma(t)$ and $\underline{\Sigma}_B^\sigma(t)$ satisfy $\overline{\Sigma}_A^{\sigma T}(t)\overline{\Sigma}_A^\sigma(t) \leq I_{nr}$, $\underline{\Sigma}_A^{\sigma T}(t)\underline{\Sigma}_A^\sigma(t) \leq I_{nr}$, $\overline{\Sigma}_B^{\sigma T}(t)\overline{\Sigma}_B^\sigma(t) \leq I_{mr}$ and, $\underline{\Sigma}_B^{\sigma T}(t)\underline{\Sigma}_B^\sigma(t) \leq I_{mr}$.

Assumption 3: There exist known constant matrices $\overline{\Delta A}_{\min}^\sigma$, $\overline{\Delta A}_{max}^\sigma$, $\underline{\Delta A}_{\min}^\sigma$ and $\underline{\Delta A}_{\max}^\sigma$ such that, for all $t \geq 0$, for all $\sigma \in \{1, 2, \dots, N\}$,

$$\begin{array}{c} \overline{\Delta A}_{\min}^\sigma \leq \overline{\Delta A}^\sigma(t) \leq \overline{\Delta A}_{\max}^\sigma \\ \underline{\Delta A}_{\min}^\sigma \leq \underline{\Delta A}^\sigma(t) \leq \underline{\Delta A}_{\max}^\sigma \end{array}$$

Similarly, there exist known constant matrices $\overline{\Delta B}_{\min}^\sigma$, $\overline{\Delta B}_{max}^\sigma$, $\underline{\Delta B}_{\min}^\sigma$ and $\underline{\Delta B}_{\max}^\sigma$ such that, for all $t \geq 0$, for all $\sigma \in \{1, 2, \dots, N\}$,

$$\begin{array}{c} \overline{\Delta B}_{\min}^\sigma \leq \overline{\Delta B}^\sigma(t) \leq \overline{\Delta B}_{\max}^\sigma \\ \underline{\Delta B}_{\min}^\sigma \leq \underline{\Delta B}^\sigma(t) \leq \underline{\Delta B}_{\max}^\sigma \end{array}$$

Assumption 3 means that the uncertainties $\overline{\Delta A}^\sigma(t)$, $\overline{\Delta B}^\sigma(t)$, $\underline{\Delta A}^\sigma(t)$ and $\underline{\Delta B}^\sigma(t)$ are unknown terms but bounded by known constant matrices. This assumption is natural when dealing with system parameter uncertainties.

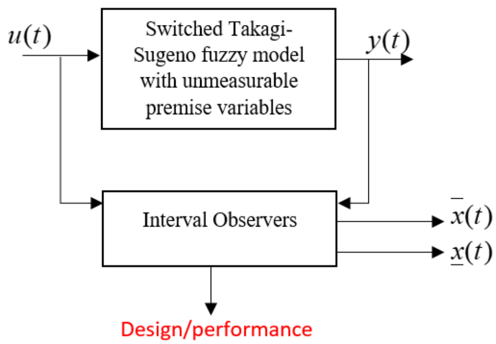

Fig. 1. A block diagram of the state estimation of the system (19) based on interval observer approach.

Remark 2: According to the design of the upper (respectively, lower) bound, first the (respectively, second) form of (19) is employed. The key idea is based on replacing unmeasurable premise variables $\mu_i^\sigma(x(t))$ by their upper or lower estimated bounds (i.e., $\mu_i^\sigma(\overline{x}(t))$ or $\mu_i^\sigma(\underline{x}(t))$) associating with respective bounded uncertainties ($\overline{\Delta A}^\sigma(t)$, $\overline{\Delta B}^\sigma(t)$) or ($\underline{\Delta A}^\sigma(t)$, $\underline{\Delta B}^\sigma(t)$).

III. Main results

In this paper, T-S interval observers are designed for a class of nonlinear switched systems described by fuzzy model-based approaches. The T-S fuzzy model can be represented in the compact form (17) with unmeasurable premise variables. Next, it is transformed into an uncertain switched T-S system (19) with upper or lower estimated premise variables. Starting from the initial state $x(0)$ which verifies $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$ and taking into account the uncertainties, an interval observer will first be *designed*. The purpose is to satisfy stability and nonnegativity properties of the observation errors. Subsequently, gains of the present interval observer will be optimized using H_∞ approach to guarantee the attenuation of additive disturbances effect in the absence of measurement noises. Hence, these gains improve the estimation accuracy by tightening the error between upper and lower bounds. Thus, the *performance* of the interval observer will be improved. The main goal of this paper is illustrated in Figure 1.

A. T-S interval observer design

The third step of this paper is to design a T-S interval observer for the switched T-S system (19). Consider the following upper and lower dynamics with $\sigma \in \{1, 2, \ldots, N\}$ and $i \in \{1, \ldots, r\}$:

$$\begin{cases} \dot{\overline{x}}(t) = \sum_{i=1}^{r} \mu_i^\sigma(\overline{x}(t))[(A_i^\sigma - L_i^\sigma C^\sigma)\overline{x}(t) + B_i^\sigma u(t) + \\ \qquad L_i^\sigma y(t) + |L_i^\sigma|\overline{V}E_p] + \overline{d} - \overline{\varphi}^\sigma_{A,\min}(t) - \overline{\varphi}^\sigma_{B,\min}(t) \\ \dot{\underline{x}}(t) = \sum_{i=1}^{r} \mu_i^\sigma(\underline{x}(t))[(A_i^\sigma - L_i^\sigma C^\sigma)\underline{x}(t) + B_i^\sigma u(t) + \\ \qquad L_i^\sigma y(t) - |L_i^\sigma|\overline{V}E_p] + \underline{d} + \underline{\varphi}^\sigma_{A,\min}(t) + \underline{\varphi}^\sigma_{B,\min}(t) \end{cases} \tag{28}$$

where

$$\overline{\varphi}^\sigma_{A,\min}(t) = \overline{\Delta A}^{\sigma+}_{\min}\underline{x}^+(t) - \overline{\Delta A}^{\sigma+}_{\max}\underline{x}^-(t) - \overline{\Delta A}^{\sigma-}_{\min}\overline{x}^+(t) + \overline{\Delta A}^{\sigma-}_{\max}\overline{x}^-(t) \tag{29}$$

$$\overline{\varphi}^\sigma_{B,\min}(t) = \overline{\Delta B}^{\sigma}_{\min}u^+(t) - \overline{\Delta B}^{\sigma}_{\max}u^-(t) \tag{30}$$

$$\underline{\varphi}^\sigma_{A,\min}(t) = \underline{\Delta A}^{\sigma+}_{\min}\underline{x}^+(t) - \underline{\Delta A}^{\sigma+}_{\max}\underline{x}^-(t) - \underline{\Delta A}^{\sigma-}_{\min}\overline{x}^+(t) + \underline{\Delta A}^{\sigma-}_{\max}\overline{x}^-(t) \tag{31}$$

$$\underline{\varphi}^\sigma_{B,\min}(t) = \underline{\Delta B}^{\sigma}_{\min}u^+(t) - \underline{\Delta B}^{\sigma}_{\max}u^-(t). \tag{32}$$

Let introduce the upper and lower observation errors $\overline{e}(t) = \overline{x}(t) - x(t)$ and $\underline{e}(t) = x(t) - \underline{x}(t)$ and note that from (19), $x(t)$ can be expressed by two different ways: (i) employing upper estimated premise variables or (ii) employing lower estimated premise variables. To obtain $\overline{e}(t)$ in (33), we use the first expression of $x(t)$ given in (19) while to obtain $\underline{e}(t)$ in (34), the second form of (19) is used. Hence,

$$\dot{\overline{e}}(t) = \sum_{i=1}^{r} \mu_i^\sigma(\overline{x}(t))(A_i^\sigma - L_i^\sigma C^\sigma)\overline{e}(t) - d(t) + \overline{\Delta A}^\sigma(t)x(t) + \overline{\Delta B}^\sigma(t)u(t) + \overline{\psi}(t) \tag{33}$$

$$\dot{\underline{e}}(t) = \sum_{i=1}^{r} \mu_i^\sigma(\underline{x}(t))(A_i^\sigma - L_i^\sigma C^\sigma)\underline{e}(t) + d(t) + \underline{\Delta A}^\sigma(t)x(t) + \underline{\Delta B}^\sigma(t)u(t) + \underline{\psi}(t) \tag{34}$$

where

$$\overline{\psi}(t) = \sum_{i=1}^{r} \mu_i^\sigma(\overline{x}(t))(L_i^\sigma v(t) + |L_i^\sigma|\overline{V}E_p) + \overline{d} - \overline{\varphi}^\sigma_{A,\min}(t) - \overline{\varphi}^\sigma_{B,\min}(t) \tag{35}$$

$$\underline{\psi}(t) = \sum_{i=1}^{r} \mu_i^\sigma(\underline{x}(t))(-L_i^\sigma v(t) + |L_i^\sigma|\overline{V}E_p) - \underline{d} - \underline{\varphi}^\sigma_{A,\min}(t) - \underline{\varphi}^\sigma_{B,\min}(t), \tag{36}$$

with $\overline{\varphi}^\sigma_{A,\min}(t)$, $\overline{\varphi}^\sigma_{B,\min}(t)$, $\underline{\varphi}^\sigma_{A,\min}(t)$ and $\underline{\varphi}^\sigma_{B,\min}(t)$ defined in (29)-(32).

A three sub-steps T-S interval estimation method is proposed as follows:

- Decouple the state $x(t)$ from the upper and lower errors dynamics (33)-(34).
- Compute appropriate gains $L_i^\sigma \in \Re^{n\times p}$ for each mode $\sigma \in \{1, 2, \ldots, N\}$ in order to establish the global asymptotic stability of the origin of the considered system (19) coupled with upper error dynamic (33) as well as of the considered system (19) coupled with lower error dynamic (34) through a Lyapunov approach. This sub-step is under the heading of "Stability property".
- Ensure the nonnegativity property of the upper and lower errors dynamics given in (33) and (34) (i.e. $\overline{e}(t) \geq 0$ and $\underline{e}(t) \geq 0$ to guarantee the relation $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$). This sub-step is under the heading of "Nonnegativity property".

We achieve the first step by defining the augmented upper and lower vectors as $\overline{e}_a(t) = \begin{bmatrix} \overline{e}^T(t) & x^T(t) \end{bmatrix}^T$ and $\underline{e}_a(t) =$

$\begin{bmatrix} \underline{e}^T(t) & x^T(t) \end{bmatrix}^T$ from which the following dynamics are obtained with $\sigma \in \{1, 2, \dots, N\}$ and $i, j \in \{1, \dots, r\}$:

$$\dot{\overline{e}}_a(t) = \sum_{i=1}^{r} \mu_i^\sigma(\overline{x}(t)) \sum_{j=1}^{r} \mu_j^\sigma(x(t))(\overline{A}_{ij}^\sigma(t)\overline{e}_a(t) + \overline{B}_{ij}^\sigma(t)u(t)) + \overline{F}d(t) + G\overline{\psi}(t) \quad (37)$$

$$\dot{\underline{e}}_a(t) = \sum_{i=1}^{r} \mu_i^\sigma(\underline{x}(t)) \sum_{j=1}^{r} \mu_j^\sigma(x(t))(\underline{A}_{ij}^\sigma(t)\underline{e}_a(t) + \underline{B}_{ij}^\sigma(t)u(t)) + \underline{F}d(t) + G\underline{\psi}(t) \quad (38)$$

where

$$\begin{cases} \overline{A}_{ij}^\sigma(t) = \begin{bmatrix} A_i^\sigma - L_i^\sigma C^\sigma & \overline{\Delta A}^\sigma(t) \\ 0 & A_j^\sigma \end{bmatrix}, \overline{B}_{ij}^\sigma = \begin{bmatrix} \overline{\Delta B}^\sigma(t) \\ B_j^\sigma \end{bmatrix} \\ \underline{A}_{ij}^\sigma(t) = \begin{bmatrix} A_i^\sigma - L_i^\sigma C^\sigma & \underline{\Delta A}^\sigma(t) \\ 0 & A_j^\sigma \end{bmatrix}, \underline{B}_{ij}^\sigma = \begin{bmatrix} \underline{\Delta B}^\sigma(t) \\ B_j^\sigma \end{bmatrix} \\ \overline{F} = \begin{bmatrix} -I \\ I \end{bmatrix}, \underline{F} = \begin{bmatrix} I \\ I \end{bmatrix}, G = \begin{bmatrix} I \\ 0 \end{bmatrix}. \end{cases}$$

The second and third steps of above-mentioned three-step T-S interval estimation method are respectively attained by the upcoming LMIs (39) and (40) of Theorem 1.

Theorem 1: Let the system (19) satisfy Assumptions 1-3 and assume that $\underline{x}(0)$, $\overline{x}(0)$ are known and the initial state $x(0)$ verifies $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$. If there exist diagonal positive matrix $P_1 \in \Re^{n\times n}$, positive definite matrix $P_2 \in \Re^{n\times n}$, matrices $K^\sigma \in \Re^{n\times p}$ and the strictly positive scalars η_i^σ, ρ_1^σ and λ^σ for all $\sigma \in \{1, \dots, N\}$ such that for all $i, j \in \{1, \dots, r\}$,

$$\begin{bmatrix} \phi_i^\sigma & 0 & P_1\mathcal{A}^\sigma \\ 0 & \Upsilon_j^\sigma & 0 \\ \mathcal{A}^{\sigma T}P_1 & 0 & -\frac{1}{\rho_1^\sigma}I \end{bmatrix} \prec 0 \quad (39)$$

$$P_1A_i^\sigma - K_i^\sigma C^\sigma + \eta_i^\sigma P_1 \geq 0 \quad (40)$$

where

$$\begin{aligned} \phi_i^\sigma &= A_i^{\sigma T}P_1 + P_1A_i^\sigma - C^{\sigma T}K_i^{\sigma T} - K_i^\sigma C^\sigma + \tfrac{3}{\lambda^\sigma}P_1 \\ K_i^\sigma &= P_1L_i^\sigma \\ \Upsilon_j^\sigma &= A_j^{\sigma T}P_2 + P_2A_j^\sigma + \tfrac{3}{\lambda^\sigma}P_2 + \rho_1^\sigma \mathrm{E}_A^T\mathrm{E}_A \end{aligned}$$

with $\mathcal{A}$, E_A defined in (21), hold, then, (28) is an interval observer for the system (17).

proof 1:

1) Stability property

Consider the following common Lyapunov function for the augmented upper dynamic (37):

$$V(\overline{e}_a(t)) = \overline{e}_a{}^T(t)P\overline{e}_a(t), \ P = diag(\begin{bmatrix} P_1 & P_2 \end{bmatrix}) \succ 0 \quad (41)$$

Taking the derivative of the Lyapunov function (63) along all trajectories of (37), then $\forall\sigma \in \{1, 2, \dots, N\}$ and $\forall i, j \in \{1, \dots, r\}$:

$$\begin{aligned} \dot{V}(\overline{e}_a(t)) &= \dot{\overline{e}}_a^T(t)P\overline{e}_a(t) + \overline{e}_a{}^T(t)P\dot{\overline{e}}_a(t) \\ &= \sum_{i=1}^{r} \mu_i^\sigma(\overline{x}(t)) \sum_{j=1}^{r} \mu_j^\sigma(x(t))(\overline{e}_a^T(t)\overline{A}_{ij}^{\sigma T}(t)P\overline{e}_a(t) \\ &\quad + \overline{e}_a^T(t)P\overline{A}_{ij}^\sigma(t)\overline{e}_a(t) + 2\overline{e}_a^T(t)P\overline{B}_{ij}^\sigma(t)u(t)) \\ &\quad + 2\overline{e}_a^T(t)P\overline{F}d(t) + 2\overline{e}_a^T(t)PG\overline{\psi}(t)) \end{aligned} \quad (42)$$

Based on Lemma 5, the following inequalities are deduced where $\lambda^\sigma > 0$ for all $\sigma \in \{1, \dots, N\}$ can be selected arbitrarily

$$\begin{aligned} 2\overline{e}_a^T(t)P\overline{B}_{ij}^\sigma(t)u(t) &\leq \tfrac{1}{\lambda^\sigma}\overline{e}_a^T(t)P\overline{e}_a(t) \\ &\quad + u^T(t)\overline{B}_{ij}^{\sigma T}(t)\left[\lambda^\sigma P\right]\overline{B}_{ij}^\sigma(t)u(t) \\ 2\overline{e}_a^T(t)P\overline{F}d(t) &\leq \tfrac{1}{\lambda^\sigma}\overline{e}_a^T(t)P\overline{e}_a(t) \\ &\quad + d^T(t)\overline{F}^T\left[\lambda^\sigma P\right]\overline{F}d(t) \\ 2\overline{e}_a^T(t)PG\overline{\psi}(t) &\leq \tfrac{1}{\lambda^\sigma}\overline{e}_a^T(t)P\overline{e}_a(t) \\ &\quad + \overline{\psi}^T(t)G^T\left[\lambda^\sigma P\right]G\overline{\psi}(t) \end{aligned} \quad (43)$$

The combination of (42) and (43) leads to:

$$\dot{V}(\overline{e}_a(t)) \leq \overline{e}_a^T(t)\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i^\sigma(\overline{x}(t))\mu_j^\sigma(x(t))\left(\overline{A}_{ij}^{\sigma T}P + P\overline{A}_{ij}^\sigma + \frac{3}{\lambda}^\sigma P\right)\overline{e}_a(t) + \upsilon^\sigma \quad (44)$$

where for $\sigma \in \{1, \dots, N\}$, $i, j \in \{1, \dots, r\}$

$$\begin{aligned} \upsilon^\sigma = & \\ &\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i^\sigma(\overline{x}(t))\mu_j^\sigma(x(t))\left(u^T(t)\overline{B}_{ij}^{\sigma T}(t)\left[\lambda^\sigma P\right]\overline{B}_{ij}^\sigma(t)u(t)\right) \\ &+ d^T(t)\overline{F}^T\left[\lambda^\sigma P\right]\overline{F}d(t) + \overline{\psi}^T(t)G^T\left[\lambda^\sigma P\right]G\overline{\psi}(t). \end{aligned} \quad (45)$$

According to the convex sum property of the weighting functions μ in (15) and from the Lyapunov-based stability analysis for the T-S fuzzy systems [52], to achieve the input-to-state stability (see Lemma 6), we need to prove that

$$\Gamma^\sigma = \overline{A}_{ij}^{\sigma T}P + P\overline{A}_{ij}^\sigma + \frac{3}{\lambda^\sigma}P \prec 0. \quad (46)$$

Recall that $P = diag(\begin{bmatrix} P_1 & P_2 \end{bmatrix})$, from (46) we have

$$\begin{aligned} \Gamma^\sigma = &\begin{bmatrix} (A_i^\sigma - L_i^\sigma C^\sigma)^T & 0 \\ \overline{\Delta A}^{\sigma T}(t) & A_j^{\sigma T} \end{bmatrix}\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \\ &+ \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\begin{bmatrix} A_i^\sigma - L_i^\sigma C^\sigma & \overline{\Delta A}^\sigma(t) \\ 0 & A_j^\sigma \end{bmatrix} \\ &+ \begin{bmatrix} \frac{3}{\lambda^\sigma}P_1 & 0 \\ 0 & \frac{3}{\lambda^\sigma}P_2 \end{bmatrix}. \end{aligned} \quad (47)$$

Then, it follows

$$\Gamma^\sigma = \begin{bmatrix} (A_i^\sigma - L_i^\sigma C^\sigma)^T P_1 + P_1(A_i^\sigma - L_i^\sigma C^\sigma) + \frac{3}{\lambda^\sigma}P_1 & P_1\overline{\Delta A}^\sigma(t) \\ \overline{\Delta A}^{\sigma T}(t)P_1 & A_j^{\sigma T}P_2 + P_2A_j^\sigma + \frac{3}{\lambda^\sigma}P_2 \end{bmatrix}. \quad (48)$$

Let rewrite (48) by separating the time-depending term $P_1\overline{\Delta A}^\sigma(t)$, we obtain

$$\Gamma^\sigma = \begin{bmatrix} (A_i^\sigma - L_i^\sigma C^\sigma)^T P_1 + P_1(A_i^\sigma - L_i^\sigma C^\sigma) + \frac{3}{\lambda^\sigma}P_1 & 0 \\ 0 & A_j^{\sigma T}P_2 + P_2A_j^\sigma + \frac{3}{\lambda^\sigma}P_2 \end{bmatrix} + \overbrace{\begin{bmatrix} 0 & P_1\overline{\Delta A}^\sigma(t) \\ \overline{\Delta A}^{\sigma T}(t)P_1 & 0 \end{bmatrix}}^{\mathcal{W}}. \quad (49)$$

The matrix $\mathcal{W}$ can be decomposed such that $\mathcal{W} = \mathcal{Q} + \mathcal{Q}^T$ where

$$\mathcal{Q} = \begin{bmatrix} 0 & P_1\overline{\Delta A}^{\sigma}(t) \\ 0 & 0 \end{bmatrix}. \quad (50)$$

Using the definition of the uncertainty $\overline{\Delta A}^{\sigma}(t)$ given in (20), it yields the following partition of $\mathcal{Q}$

$$\mathcal{Q} = \overbrace{\begin{bmatrix} P_1\mathcal{A}^{\sigma} & 0 \\ 0 & 0 \end{bmatrix}}^{X} \overbrace{\begin{bmatrix} 0 & \overline{\Sigma}_A^{\sigma}(\mathrm{t})\mathrm{E}_A \\ 0 & 0 \end{bmatrix}}^{Y}. \quad (51)$$

Choosing $\Omega = diag([\underbrace{\rho_1^{\sigma}\dots\rho_1^{\sigma}}_{n \text{ terms}}\underbrace{\rho_2^{\sigma}\dots\rho_2^{\sigma}}_{n \text{ terms}}]) \succ 0$ with ρ_1^{σ}, ρ_2^{σ} are any strictly positive scalars for all $\sigma \in \{1,\dots,N\}$. Applying Lemma 4 to (51) yields

$$\mathcal{W} \leq X\Omega^{-1}X^T + Y^T\Omega Y. \quad (52)$$

Bearing in mind that $\overline{\Sigma}_A^{\sigma T}(t)\overline{\Sigma}_A^{\sigma}(t) \leq I_{nr}$ (see Remark 1), the following inequality holds

$$\mathcal{W} \leq diag([\ \tfrac{1}{\rho_1^{\sigma}}P_1\mathcal{A}^{\sigma}\mathcal{A}^{\sigma T}P_1 \quad \rho_1^{\sigma}\mathrm{E}_A^T\mathrm{E}_A\ ]) \quad (53)$$

Substituting (53) in (49) leads to:

$$\Gamma^{\sigma} = diag([\ \Xi_i^{\sigma} \quad \Upsilon_j^{\sigma}\ ]) \quad (54)$$

where

$$\Xi_i^{\sigma} = (A_i^{\sigma} - L_i^{\sigma}C^{\sigma})^T P_1 + P_1(A_i^{\sigma} - L_i^{\sigma}C^{\sigma}) + \frac{3}{\lambda^{\sigma}}P_1 + \frac{1}{\rho_1^{\sigma}}P_1\mathcal{A}^{\sigma}\mathcal{A}^{\sigma T}P_1$$

$$\Upsilon_j^{\sigma} = A_j^{\sigma T}P_2 + P_2A_j^{\sigma} + \frac{3}{\lambda^{\sigma}}P_2 + \rho_1^{\sigma}\mathrm{E}_A^T\mathrm{E}_A$$

From LMI (39), based on the Schur complement [47] with $K_i^{\sigma} = P_1L_i^{\sigma}$ we can conclude that $\Gamma^{\sigma} \prec 0$ for all $\sigma \in \{1,\dots,N\}$. On the other hand, because the state $x(t)$ and measurement noises $v(t)$ are all bounded in norm (see Assumptions 1-2 which are accepted in many applications), $\overline{\psi}(t)$ given in (35) is bounded. Moreover the known input $u(t)$ and additive disturbances $d(t)$ are also bounded in norm as stated in Assumption 1 and 2, it follows that v^{σ} in (44) is bounded for all $\sigma \in \{1,\dots,N\}$. Thus, from (44) the augmented upper dynamic (37) is ISS due to Lemma 6. Similarly one can prove that the augmented lower dynamic (38) is ISS. Hence, the stability property of the interval observer (28) can be deduced since $|\overline{x} - \underline{x}| \leq |\underbrace{\overline{x} - x}_{=\overline{e}}| + |\underbrace{x - \underline{x}}_{=\underline{e}}|$.

2 Nonnegativity property

First, from (29)-(32) and Lemma 2, the following inequalities hold

$$\begin{aligned} &\overline{\Delta A}^{\sigma}(t)x(t) \geq \overline{\varphi}_{A,\min}^{\sigma}(t),\ \underline{\Delta A}^{\sigma}(t)x(t) \geq \underline{\varphi}_{A,\min}^{\sigma}(t) \\ &\overline{\Delta B}^{\sigma}(t)u(t) \geq \overline{\varphi}_{B,\min}^{\sigma}(t),\ \underline{\Delta B}^{\sigma}(t)u(t) \geq \underline{\varphi}_{B,\min}^{\sigma}(t) \end{aligned} \quad (55)$$

From Assumption 1, we have for all $\sigma \in \{1,\dots,N\}$, $i \in \{1,\dots,r\}$, $\overline{d} - d \geq 0$, $d - \underline{d} \geq 0$, $L_i^{\sigma}v(t) + |L_i^{\sigma}|\overline{V}E_p \geq 0$ and $-L_i^{\sigma}v(t) + |L_i^{\sigma}|\overline{V}E_p \geq 0$. Thus from (35)-(36), it holds that

$$\begin{aligned} &\overline{\psi}(t) - d(t) + \overline{\Delta A}^{\sigma}(t)x(t) + \overline{\Delta B}^{\sigma}(t)u(t) \geq 0 \\ &d(t) + \underline{\Delta A}^{\sigma}(t)x(t) + \underline{\Delta B}^{\sigma}(t)u(t) + \underline{\psi}(t) \geq 0 \end{aligned} \quad (56)$$

Subsequently, thanks to (40) and Lemma 3, one can ensure that $(A_i^{\sigma} - L_i^{\sigma}C^{\sigma})$ is Metzler for all $\sigma \in \{1,\dots,N\}$, for all $i \in \{1,\dots,r\}$ since $P_1(A_i^{\sigma} - L_i^{\sigma}C^{\sigma}) + \eta_i^{\sigma}P_1 \geq 0, \forall\sigma \in \{1,\dots,N\}, \forall i \in \{1,\dots,r\}$ and P_1 is diagonal positive matrix.

Lastly, according to Lemma 1 and bearing in mind the Metzler property of $(A_i^{\sigma} - L_i^{\sigma}C^{\sigma})$ together with (56), if $\overline{x}(0)$ and $\underline{x}(0)$ are supposed to be known such that

$$\begin{cases} \overline{e}(0) = \overline{x}(0) - x(0) \geq 0 \\ \underline{e}(0) = x(0) - \underline{x}(0) \geq 0 \end{cases},$$

then the dynamics of the estimation errors given in (33)-(34) stay positive and consequently, $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ which completes the proof.

B. Optimal T-S interval design via H_{∞} approach with respect to additive disturbances

In this section, we consider that measurement noises are not present (i.e., $v(t) = 0$). The main objective is to adapt the design of the T-S interval observer (28) in order to estimate an ultimate-bound guaranteeing a tighter interval width which represents the fourth step of this paper. In fact, this section is devoted to the computation of gains L_i^{σ}, $\sigma = \{1,\dots,N\}$, $i \in \{1,\dots,r\}$ to ensure stability properties. Notice that these gains decide also the tightness of the interval width. Hence, the goal is not only to ensure the stability of the ultimate-bound as in Section III-A but also to improve the accuracy of the T-S interval observer (28) and the idea is based on an H_{∞} approach. The effect of the known bound of the additive disturbance d on the estimation error is reduced by the observer gain matrices. In other words, we are interested in computing observer gains L_i^{σ} which minimize a cost function to be introduced later.

Given the structure of the T-S interval observer (28), by taking account $\overline{V} = v(t) = 0$, (33)-(34) can be rewritten as follows

$$\dot{\overline{e}}(t) = \sum_{i=1}^{r}\mu_i^{\sigma}(\overline{x}(t))((A_i^{\sigma} - L_i^{\sigma}C^{\sigma})\overline{e}(t) + \overline{d} - d(t) + \overline{\Delta A}^{\sigma}(t)x(t) + \overline{\Delta B}^{\sigma}(t)u(t) + \overline{\psi}^*(t) \quad (57)$$

$$\dot{\underline{e}}(t) = \sum_{i=1}^{r}\mu_i^{\sigma}(\underline{x}(t))((A_i^{\sigma} - L_i^{\sigma}C^{\sigma})\underline{e}(t) + d(t) - \underline{d} + \underline{\Delta A}^{\sigma}(t)x(t) + \underline{\Delta B}^{\sigma}(t)u(t) + \underline{\psi}^*(t) \quad (58)$$

where $\overline{\psi}^*(t) = -\overline{\varphi}_{A,\min}^{\sigma}(t) - \overline{\varphi}_{B,\min}^{\sigma}(t)$ and $\underline{\psi}^*(t) = -\underline{\varphi}_{A,\min}^{\sigma}(t) - \underline{\varphi}_{B,\min}^{\sigma}(t)$ with $\overline{\varphi}_{A,\min}^{\sigma}(t)$, $\overline{\varphi}_{B,\min}^{\sigma}(t)$, $\underline{\varphi}_{A,\min}^{\sigma}(t)$ and $\underline{\varphi}_{B,\min}^{\sigma}(t)$ defined in (29)-(32).

Analogously to (37)-(38) by defining $\overline{e}_a(t) = \begin{bmatrix} \overline{e}^T(t) & x^T(t) \end{bmatrix}^T$ and $\underline{e}_a(t) = \begin{bmatrix} \underline{e}^T(t) & x^T(t) \end{bmatrix}^T$, the following dynamics are obtained with $\sigma \in \{1,2,\dots,N\}$ and $i,j \in \{1,\dots,r\}$:

$$\dot{\overline{e}}_a(t) = \sum_{i=1}^{r}\mu_i^{\sigma}(\overline{x}(t))\sum_{j=1}^{r}\mu_j^{\sigma}(x(t))(\overline{A}_{ij}^{\sigma}(t)\overline{e}_a(t) + \overline{B}_{ij}^{\sigma}(t)u(t)) + \overline{E}\overline{d} + \overline{F}d(t) + G\overline{\psi}^*(t) \quad (59)$$

$$\underline{\dot{e}}_a(t) = \sum_{i=1}^{r} \mu_i^\sigma(\underline{x}(t)) \sum_{j=1}^{r} \mu_j^\sigma(x(t))(\underline{A}_{ij}^\sigma(t)\underline{e}_a(t) + \underline{B}_{ij}^\sigma(t)u(t)) + \underline{F}d(t) + \underline{E}\underline{d} + G\underline{\psi}^*(t) \quad (60)$$

where $\overline{E} = \begin{bmatrix} I & 0 \end{bmatrix}^T$, $\underline{E} = \begin{bmatrix} -I & 0 \end{bmatrix}^T$, $\overline{F} = \begin{bmatrix} -I & I \end{bmatrix}^T$, $\underline{F} = \begin{bmatrix} I & I \end{bmatrix}^T$.

Theorem 2: Let the system (19) satisfy Assumptions 1-3 and assume that $\underline{x}(0)$, $\overline{x}(0)$ are known and the initial state $x(0)$ verifies $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$. If there exist diagonal positive matrix $P_1 \in \Re^{n\times n}$, positive definite matrix $P_2 \in \Re^{n\times n}$, matrices $K^\sigma \in \Re^{n\times p}$ and the strictly positive scalars η_i^σ, ρ_1^σ, $\overline{\gamma}$ and λ^σ for all $\sigma \in \{1,\ldots,N\}$ such that for all $i,j \in \{1,\ldots,r\}$, the following constrained minimization problem

$$\begin{aligned} &\underset{P_1,P_2,K_i^\sigma,\rho_1^\sigma}{\text{minimize}} \quad \overline{\gamma} \\ &\text{subject to} \quad \begin{bmatrix} \phi_i^\sigma & 0 & P_1 & P_1\mathcal{A}^\sigma \\ 0 & \Upsilon_j^\sigma & 0 & 0 \\ P_1 & 0 & -\overline{\gamma}I & 0 \\ \mathcal{A}^{\sigma T}P & 0 & 0 & -\frac{1}{\rho_1^\sigma}I \end{bmatrix} \prec 0 \\ &\qquad P_1A_i^\sigma - K_i^\sigma C^\sigma + \eta_i^\sigma P_1 \geq 0. \end{aligned} \quad (61)$$

where

$$\begin{aligned} \phi_i^\sigma &= A_i^{\sigma T}P_1 + P_1A_i^\sigma - C^{\sigma T}K_i^{\sigma T} - K_i^\sigma C^\sigma + \tfrac{3}{\lambda^\sigma}P_1 + I_n \\ K_i^\sigma &= P_1 L_i^\sigma \\ \Upsilon_j^\sigma &= A_j^{\sigma T}P_2 + P_2A_j^\sigma + \tfrac{3}{\lambda^\sigma}P_2 + \rho_1^\sigma \mathrm{E}_A^T\mathrm{E}_A, \end{aligned}$$

with $\mathcal{A}$, E_A defined in (21), is solved, then when $v(t) = \overline{V} = 0$, (28) is an optimal interval observer for the system (17) that guarantees the attenuation of additive disturbances effect with the cost function computed by

$$\gamma = \sqrt{\overline{\gamma}}. \quad (62)$$

Remark 3: Notice that the terms η_i^σ are fixed before solving the LMIs (40) and (61). Thus, those LMIs are not nonlinear optimization problems.

proof 2:

1) Stability property

Employing the following common Lyapunov function for the augmented upper dynamic (59):

$$V(\overline{e}_a(t)) = \overline{e}_a{}^T(t)P\overline{e}_a(t),\ P = diag(\begin{bmatrix} P_1 & P_2 \end{bmatrix}) \succ 0 \quad (63)$$

to prove that (59) is ISS. Indeed, in the same way of the proof of Theorem 1 given in (42)-(44), it follows that

$$\dot{V}(\overline{e}_a(t)) \leq \overline{e}_a^T(t)\left(\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i^\sigma(\overline{x}(t))\mu_j^\sigma(x(t))\Gamma^\sigma\right)\overline{e}_a(t) + \overline{d}^T\bar{E}^TP\overline{e}_a(t) + \overline{e}_a^TP\bar{E}\bar{d} + \vartheta^\sigma \quad (64)$$

where

$$\vartheta^\sigma = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i^\sigma(\overline{x}(t))\mu_j^\sigma(x(t))\left(u^T(t)\overline{B}_{ij}^{\sigma T}(t)\left[\lambda^\sigma P\right]\overline{B}_{ij}^\sigma(t)u(t)\right) + d^T(t)\overline{F}^T\left[\lambda^\sigma P\right]\overline{F}d(t) + \overline{\psi}^{*T}(t)G^T\left[\lambda^\sigma P\right]G\overline{\psi}^*(t) \quad (65)$$

with $\overline{\psi}^*$ defined in (57).

Besides, the upper estimation error given in (57) can be seen as

$$\overline{e}(t) = H\overline{e}_a(t) \quad (66)$$

where $H = \begin{bmatrix} I & 0 \end{bmatrix}$.

Based on H_∞ approach, the optimal gains are computed in order to minimize the following cost function $\forall\sigma \in \{1,2,\ldots,N\}$ and $\forall i,j \in \{1,\ldots,r\}$:

$$\begin{aligned} &\underset{L_i^\sigma \in \Re^{n\times p}}{\text{minimize}} \quad \gamma^2 \\ &\text{subject to} \quad \frac{\|\overline{e}(t)\|_2}{\|\overline{d}\|_2} \leq \gamma^2 \end{aligned} \quad (67)$$

The effect of the known bound of the additive disturbances $d(t)$ on the upper observation error $\overline{e}$ is reduced by the observer gain matrix L_i^σ which are computed via the minimization of the positive real number $\overline{\gamma} = \gamma^2$.

The optimization problem (67) can be reformulated under an LMI form using the bounded-real lemma [47] for (57) which leads to:

$$\dot{V}(\overline{e}_a(t)) + \overline{e}^T(t)\overline{e}(t) - \overline{\gamma}\overline{d}^T\overline{d} \leq 0 \quad (68)$$

The same reasoning of the proof of Theorem 1 applies, we deduce that ϑ^σ given in (65) is bounded. Thus, by substituting (64) and (66) in (68), (59) is ISS if the following inequality holds

$$\begin{aligned} &\overline{e}_a^T(t)\Gamma^\sigma\overline{e}_a(t) + \overline{d}^T\overline{E}^TP\overline{e}_a(t) + \overline{e}_a^TP\overline{E}\overline{d} \\ &+\overline{e}_a^T(t)H^TH\overline{e}_a - \overline{\gamma}\overline{d}^T\overline{d} \leq 0 \end{aligned} \quad (69)$$

or equivalently

$$\begin{bmatrix} \overline{e}_a \\ \overline{d} \end{bmatrix}^T \begin{bmatrix} \Gamma^\sigma + H^TH & P\overline{E} \\ \overline{E}^TP & -\overline{\gamma}I \end{bmatrix} \begin{bmatrix} \overline{e}_a \\ \overline{d} \end{bmatrix} \leq 0 \quad (70)$$

Replacing the term Γ^σ by its expression given in (54) and from the definition of H in (66) we have $H^TH = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \in \Re^{2n\times 2n}$, the inequality (70) holds if the subsequent one is satisfied

$$\begin{bmatrix} \Xi_i^\sigma + I_n & 0 & P\overline{E} \\ 0 & \Upsilon_j^\sigma & 0 \\ \overline{E}^TP & 0 & -\overline{\gamma}I \end{bmatrix} \prec 0 \quad (71)$$

From the LMI (61), based on the Schur complement [47] with $K_i^\sigma = P_1L_i^\sigma$, this allows us to conclude. Similarly one can prove that the augmented lower dynamic (60) is ISS. Hence, the stability property of the interval observer (28) can be deduced since $|\overline{x} - \underline{x}| \leq |\underbrace{\overline{x} - x}_{=\overline{e}}| + |\underbrace{x - \underline{x}}_{=\underline{e}}|$.

2 Nonnegativity property

This proof is identical to the one given in Theorem 1.

Remark 4:

- LMIs given in Theorem 1 and Theorem 2 ensure at the same time the *ISS* and nonnegativity properties of T-S interval observer (28). In addition, minimizing cost function in Theorem 2 yields optimal gains that improve the accuracy of the present T-S interval observer.

- The presence of measurement noises in (61) leads to a nonlinear optimization problem which cannot be easily solved. This is the design constraints of H_∞-based T-S interval observer (28) . We conjecture that one can overcome these design constraints by proposing another guaranteed state estimation approach, e.g., zonotopic technique paired with H_∞ design which is an interesting perspective for the future work. Furthermore, it is worth highlighting that in the presence of $v(t)$, Theorem 1 works and T-S interval observer (28) can be always designed.

C. Application to robust fault detection

In this section, the previous results are used to generate residuals for fault detection. Under the presence of a sensor fault and the absence of measurement noises ($v(t) = 0$), system (17) can be represented by:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \mu_i^\sigma(x(t))(A_i^\sigma x(t) + B_i^\sigma u(t)) + d(t) \\ y(t) = C^\sigma x(t) + f(t) \\ \forall\sigma \in \{1, 2, \dots, N\}, \forall i \in \{1, \dots, r\}, \end{cases} \tag{72}$$

where $f(t) \in \Re^p$ denotes p^{th} sensor fault. The principle is to compare the measurements $y(t)$ with their estimates $\widehat{y}(t)$ provided by a faultless model. The comparison leads to the generation of a residual $r(t) \in \Re^p$ given by:

$$r(t) = \widehat{y}(t) - y(t). \tag{73}$$

In a fault-free operation, the residual are around zero. Nevertheless, when considering a system affected by perturbations and uncertainties given in (19), the residuals deviate from zero even in the fault-free scenario. To cope with this problem, a passive approach is used based on the interval observer (28) designed in the previous section.

Based on (2) of *Lemma* 2, the lower and upper outputs of the system (17) are given by:

$$\begin{cases} \overline{y}(t) = C^{\sigma+}\overline{x}(t) - C^{\sigma-}\underline{x}(t) \\ \underline{y}(t) = C^{\sigma+}\underline{x}(t) - C^{\sigma-}\overline{x}(t) \end{cases} \tag{74}$$

Let $\left[\underline{y}(t), \overline{y}(t)\right]$ be the domain of the output $y(t)$, the fault detection test can be formulated as $y(t) \notin \left[\underline{y}(t), \overline{y}(t)\right]$ which is equivalent to:

$$0 \notin \left[\underline{r}(t), \overline{r}(t)\right]. \tag{75}$$

where

$$\begin{cases} \underline{r}(t) = \underline{y}(t) - y(t) = -C^{\sigma+}(x(t) - \underline{x}(t)) \\ \qquad\qquad\qquad\qquad -C^{\sigma-}(\overline{x}(t) - x(t)) \\ \overline{r}(t) = \overline{y}(t) - y(t) = C^{\sigma+}(\overline{x}(t) - x(t)) \\ \qquad\qquad\qquad\qquad -C^{\sigma-}(x(t) - \underline{x}(t)) \end{cases} \tag{76}$$

Thus, the residual is described by an adaptive threshold.

Remark 5: The considered system in this paper is affected by unknown but bounded disturbances. If these bounds are large, so does the width of the interval observer and it may lead to misdetection of small faults. The proposed T-S interval observer design method given in (28) allows to compute optimal gains which attenuate the effect of the system's disturbances and ensure a tighter interval width which make it possible to detect low magnitude faults.

IV. Numerical simulations

A. Example 1

In order to illustrate the effectiveness of the proposed approaches, Theorem 1 is firstly illustrated and secondly, a comparative study is made between the results obtained in Theorem 1 and those optimized in Theorem 2 in the absence of measurement noises.

Let consider a switched system characterized by two nonlinear modes (i.e. $\sigma \in \{1, 2\}$) where each mode is represented by T-S fuzzy models with two local models (i.e. $r = 2$):

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2} \mu_i^\sigma(\xi(t))(A_i^\sigma x(t) + B_i^\sigma u(t)) + d(t) \\ y(t) = C^\sigma x(t) + v(t) \end{cases} \tag{77}$$

where

For Mode $\sigma = 1$:

$$A_1^1 = \begin{bmatrix} -1.51 & -0.262 \\ 0 & -0.1 \end{bmatrix}, \ A_2^1 = \begin{bmatrix} -0.86 & 1.47 \\ 0 & -0.15 \end{bmatrix} \\ B_1^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ B_2^1 = B_1^1, \ C^1 = \begin{bmatrix} 0 & 1.5 \end{bmatrix} \tag{78}$$

For Mode $\sigma = 2$:

$$A_1^2 = \begin{bmatrix} -5.55 & 4.1 \\ 0 & -0.1 \end{bmatrix}, \ A_2^2 = \begin{bmatrix} -2.65 & 2.34 \\ 0 & -0.15 \end{bmatrix} \\ B_1^2 = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}, \ B_2^2 = B_1^2, \ C^2 = \begin{bmatrix} 0 & 1.5 \end{bmatrix} \tag{79}$$

The weighting functions are hyperbolic tangent functions and depend on the state of the switched system such as

$$\begin{cases} \xi(t) = x_1(t) \\ \mu_1^\sigma(x(t)) = \frac{1}{2}(1 - \tanh(x_1(t)), \ \forall\sigma \in \{1, 2\} \\ \mu_2^\sigma(x(t)) = 1 - \mu_1^\sigma(x_1(t)), \ \forall\sigma \in \{1, 2\} \end{cases} \tag{80}$$

For the simulation, the disturbances and the measurement noises are chosen such as:

$$\begin{array}{l} d(t) = 0.1\begin{bmatrix} \sin(2.7t) & \cos(2.7t) \end{bmatrix}^T \\ \overline{d} = -\underline{d} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}^T \\ v(t) = 0.01\sin(2.7t), \ \overline{V} = 0.01 \end{array} \tag{81}$$

Thus, Assumption 1 is satisfied. The switching signal between the two modes of the considered system is plotted in Figure 2.

The initial conditions are $x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and $\overline{x}(0) = -\underline{x}(0) = \begin{bmatrix} 0.05 & 0.02 \end{bmatrix}^T$. In the simulation, the control input is given by $u(t) = 0.1\sin(2.7t)$. The solutions of Theorem 1 and 2 are obtained for $\lambda^1 = 25.96$, $\lambda^2 = 85.76$, $\eta_1^1 = \eta_2^1 = 10$ and $\eta_1^2 = \eta_2^2 = 25$.

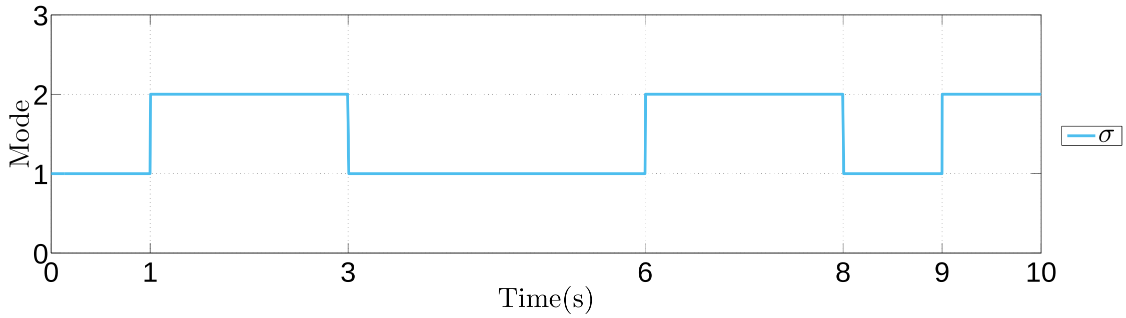

Fig. 2. Switching signal σ

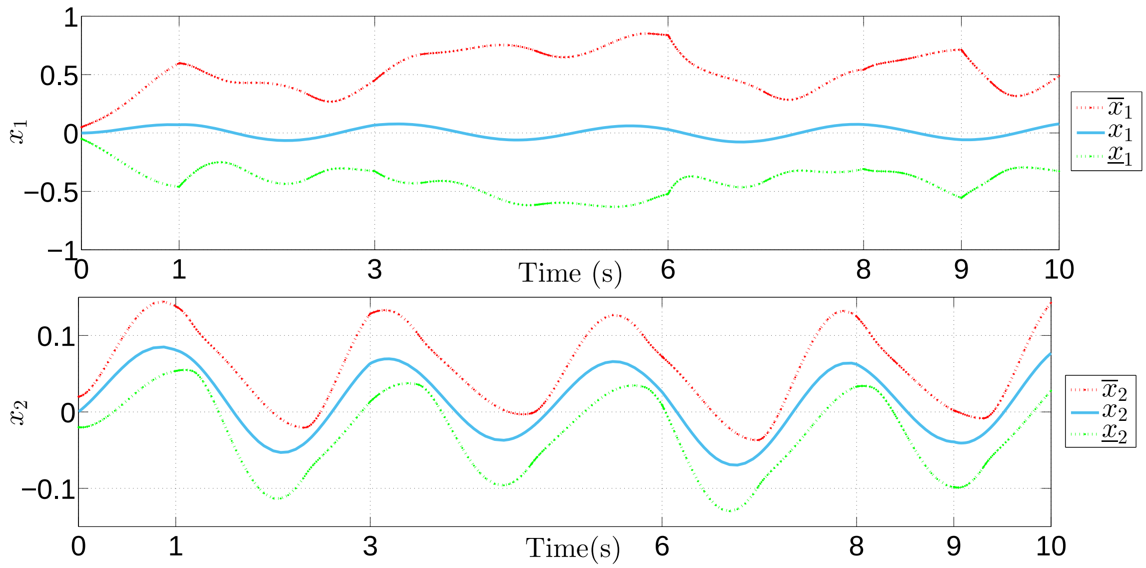

Fig. 3. x_1, x_2 and their interval estimations with measurement noises $v(t)$ by Theorem 1

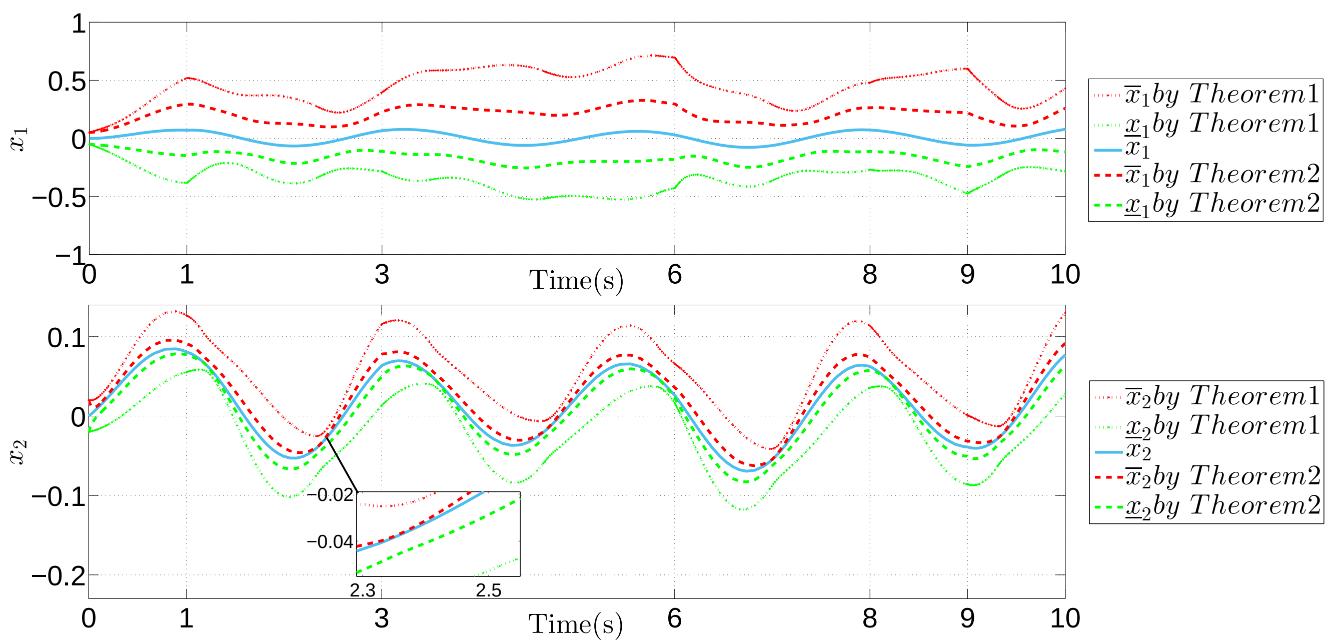

Fig. 4. x_1, x_2 and their interval estimations without measurement noises ($v(t) = 0$) by Theorem 1 and by Theorem 2

1) Solution of Theorem 1: Using the package Yalmip toolbox [53], the solution of LMIs (39) of Theorem 1 is given by:

$$
\begin{aligned}
L_1^1 &= \begin{bmatrix} -3.0735 \\ 2.5767 \end{bmatrix}, L_2^1 = \begin{bmatrix} -0.8078 \\ 2.5981 \end{bmatrix}, \rho_1^1 = 0.2345, \\
L_1^2 &= \begin{bmatrix} -2.4280 \\ 2.4189 \end{bmatrix}, L_2^2 = \begin{bmatrix} -0.8353 \\ 2.5057 \end{bmatrix}, \rho_1^2 = 0.6645, \\
P_1 &= \begin{bmatrix} 0.0425 & 0.0000 \\ 0.0000 & 0.1878 \end{bmatrix}, P_2 = \begin{bmatrix} 0.4463 & 0.0000 \\ 0.0000 & 15.1907 \end{bmatrix}.
\end{aligned}
$$

We verify that the matrices $A_i^\sigma - L_i^\sigma C^\sigma$ are Metzler $\forall \sigma \in \{1,2\}$, $\forall i \in \{1,2\}$. The T-S interval observer (19) is applied. The simulation results given in Figure 3 show that the state stays in the estimated interval all the time, even when the measurement noises and the additive disturbances in (81) are present. In addition, the upper and lower bounds remain stable despite the switching instants. However here the optimization issue of interval width between upper and lower bound is not considered.

2) Comparisons study between Theorems 1 and 2: In this section, we consider that the system is affected only by additive disturbances $d(t)$ (i.e. $v(t) = 0$). Using the package CVX [54], the solution of LMIs (61) of Theorem 2 is given by

$$
\begin{aligned}
L_1^1 &= \begin{bmatrix} -0.1747 \\ 10.1468 \end{bmatrix}, L_2^1 = \begin{bmatrix} -0.9800 \\ 10.1135 \end{bmatrix}, \rho_1^1 = 1.3062\ 10^6 \\
L_1^2 &= \begin{bmatrix} 2.7333 \\ 10.1334 \end{bmatrix}, L_2^2 = \begin{bmatrix} 1.5600 \\ 10.1000 \end{bmatrix}, \rho_1^2 = 3.5739\ 10^6 \\
P_1 &= \begin{bmatrix} 1.2465 & 0.0000 \\ 0.0000 & 32.8409 \end{bmatrix}, \\
P_2 &= 10^7 \begin{bmatrix} 0.2379 & 0.0000 \\ 0.0000 & 8.3253 \end{bmatrix}.
\end{aligned}
$$

The resulting attenuation level is $\overline{\gamma} = 1.2465$. We verify that the matrices $A_i^\sigma - L_i^\sigma C^\sigma$ are Metzler $\forall \sigma \in \{1,2\}$, $\forall i \in \{1,2\}$.

For the purpose of comparison, the simulation of the T-S interval observer (28) with non optimal and optimal gains are depicted in Figure 4 where solid lines present the state and dashed lines present the estimated bounds. Under the same simulation conditions, the results show that the gains computed by Theorem 2 give more accurate interval estimation than the ones computed by Theorem 1 in the absence of measurement noise. Optimal gains offer a tighter interval width. This is explained by the reduction of the effect of the known bound of the additive disturbances on the estimation error. The accuracy of interval observer is then improved.

B. Example 2

To show the effectiveness of the proposed fault detection method, a switched system described by (72) is considered as follows

$$
\begin{aligned}
A_1^1 &= \begin{bmatrix} -0.9 & 0 & -0.45 \\ 0 & -2.1 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}, \\
A_2^1 &= \begin{bmatrix} -3.86 & 0 & 1.22 \\ 0 & -0.15 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}, \\
A_1^2 &= \begin{bmatrix} -5.5 & 0 & 1.5 \\ 0 & -1.1 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}, \\
A_2^2 &= \begin{bmatrix} -2.6 & 0 & 0.3 \\ 0 & -0.15 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}, \\
B_1^1 &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \\
B_2^1 &= B_1^2 = B_2^2 = B_1^1, \\
C^1 &= \begin{bmatrix} 0 & 0 & 1.2 \end{bmatrix}, C^2 = \begin{bmatrix} 0 & 0 & 1.7 \end{bmatrix}.
\end{aligned}
$$

The weighting functions are hyperbolic tangent functions and depend on the unmeasured state x_1:

$$
\begin{cases} \xi(t) = x_1(t) \\ \mu_1^\sigma(x(t)) = \frac{1}{2}(1 - \tanh(x_1(t))),\ \forall \sigma \in \{1,2\} \\ \mu_2^\sigma(x(t)) = 1 - \mu_1^\sigma(x_1(t)),\ \forall \sigma \in \{1,2\} \end{cases} \tag{82}
$$

For the simulation, the disturbances are chosen such as: $d(t) = 0.1\begin{bmatrix} \cos(3.5t) & \cos(3.5t) & \cos(3.5t) \end{bmatrix}^T$ and $\overline{d} = -\underline{d} = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}^T$. Thus, Assumption 1 is satisfied. The switching signal between the two modes of the considered system is plotted similarly to Example IV-A in Figure 2. The fault signal is set up as:

$$
f(t) = \begin{cases} 0.03,\ 2s \le t \le 4s \\ 0.02,\ 8s \le t \le 9s \\ 0 \text{ otherwise} \end{cases} \tag{83}
$$

The initial conditions are $x(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ and $\overline{x}(0) = -\underline{x}(0) = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}^T$. By fixing $\lambda^1 = 85.96$, $\lambda^2 = 85.76$, $\eta_1^1 = \eta_2^1 = 10$ and $\eta_1^2 = \eta_2^2 = 26$, the solution of LMIs (61) of Theorem 2 are obtained using the package CVX [54]. The values of the optimal gains are given by:

$$
\begin{aligned}
L_1^1 &= \begin{bmatrix} -0.3750 \\ 0.0000 \\ 16.4867 \end{bmatrix}, L_2^1 = \begin{bmatrix} 1.0167 \\ 0.0000 \\ 16.4867 \end{bmatrix}, \\
L_1^2 &= \begin{bmatrix} 0.0824 \\ 0.0000 \\ 11.6377 \end{bmatrix}, L_2^2 = \begin{bmatrix} 0.1765 \\ 0.0000 \\ 11.6377 \end{bmatrix}
\end{aligned}
$$

The attenuation level is $\overline{\gamma} = 7.5443$. The matrices $A_i^\sigma - L_i^\sigma C^\sigma$ are Metzler for all $\sigma \in \{1,2\}$ and for all $i \in \{1,2\}$. In Figure 5, it is clear that the relations $\underline{y}(t) \le y(t) \le \overline{y}(t)$ and $0 \in [\underline{r}(t), \overline{r}(t)]$ hold in fault-free case while these relations are broken when the fault occurs. It should be noted that despite the low values of the considered fault (83), the detection is successful. At the instant $t = 4.01s$, the fault is still detected, which is explained by the fault extension since the switching instant $t = 3s$ happens during the faulty period $2s \le t \le 4s$.

V. Conclusion

In this paper, a new approach is proposed to deal with the problem of interval observer design for nonlinear switched systems described by T-S fuzzy models. The problem is challenging because the premise variables are unmeasurable and the membership functions μ_i, $i \in \{1, \ldots, r\}$ are not assumed to be globally Lipschitz. Thanks to the fact that the premise variables can be replaced by their interval estimates, this structure allows us to see the considered system as an uncertain system subject to unknown but bounded disturbances/measurement noises. The first goal of the paper is to design T-S interval observer by solving a set of LMIs that ensure at the same time the stability and nonnegativity properties. In the absence of measurement noises, the second goal of the paper is to tighten the width of the present interval observer by pairing it with H_∞ approach. A sensor fault detection scheme based on interval observers is also given and can be considered as an interesting application. A comparative study between the first and second contribution as well as fault detection issues are successfully done by simulations to illustrate the efficiency of the proposed approaches. Future works will be focused on improving these results when the switching signal is unknown and has to be estimated. On the other hand, the design of closed-loop interval observers which not only gives interval estimates, but also helps to stabilize possibly unstable plants by feedback and adaptive controllers in the spirit of what is done and reviewed in [55]–[57], is also an interesting perspective.

References

[1] Arjan J Van Der Schaft and Johannes Maria Schumacher. *An introduction to hybrid dynamical systems*, volume 251. Springer London, 2000.

[2] Rafal Goebel, Ricardo G Sanfelice, and Andrew R Teel. Hybrid dynamical systems. *IEEE Control Systems*, 29(2):28–93, 2009.

[3] Michael S Branicky. Multiple lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on automatic control*, 43(4):475–482, 1998.

[4] Chongyang Liu, Zhaohua Gong, Enmin Feng, and Hongchao Yin. Optimal switching control of a fed-batch fermentation process. *Journal of Global Optimization*, 52(2):265–280, 2012.

[5] Neil B Petroff. *Biomimetic sensing for robotic manipulation*. University of Notre Dame, 2007.

[6] Wei Zhang, Jianghai Hu, and Yung-Hsiang Lu. Optimal power modes scheduling using hybrid systems. In *Proceedings of the American Control Conference, New York City, NY*, pages 2781–2786, 2007.

[7] Yury Orlov. Finite time stability and robust control synthesis of uncertain switched systems. *SIAM Journal on Control and Optimization*, 43(4):1253–1271, 2004.

[8] Dong Wang, Wei Wang, and Peng Shi. Robust fault detection for switched linear systems with state delays. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 39(3):800–805, 2009.

[9] Francisco Javier Bejarano and Alessandro Pisano. Switched observers for switched linear systems with unknown inputs. *IEEE Transactions on Automatic Control*, 56(3):681–686, 2011.

[10] Xudong Zhao, Hao Liu, Junfeng Zhang, and Hongyi Li. Multiple-mode observer design for a class of switched linear systems. *IEEE Transactions on Automation Science and Engineering*, 12(1):272–280, 2015.

[11] Héctor Ríos, Denis Efimov, Jorge Davila, Tarek Raissi, Leonid Fridman, and Ali Zolghadri. State estimation for linear switched systems with unknown inputs. *IFAC Proceedings Volumes*, 45(9):271–276, 2012.

[12] Jeremy Van Gorp, Michael Defoort, Kalyana C. Veluvolu, and Mohamed Djemai. Hybrid sliding mode observer for switched linear systems with unknown inputs. *Journal of the Franklin Institute*, 351(7):3987–4008, 2014.

[13] Panos J Antsaklis. A brief introduction to the theory and applications of hybrid systems. In *Proc IEEE, Special Issue on Hybrid Systems: Theory and Applications*, pages 879–887. Citeseer, 2000.

[14] Ryan C Loxton, Kok Lay Teo, Volker Rehbock, and WK Ling. Optimal switching instants for a switched-capacitor dc/dc power converter. *Automatica*, 45(4):973–980, 2009.

[15] Wen-An Zhang, Li Yu, and Shu Yin. A switched system approach to H_∞ control of networked control systems with time-varying delays. *Journal of the Franklin Institute*, 348(2):165–178, 2011.

[16] Tomohiro Takagi and Michio Sugeno. Fuzzy identification of systems and its applications to modeling and control. In *Readings in Fuzzy Sets for Intelligent Systems*, pages 387–403. Elsevier, 1993.

[17] Yong-Yan Cao and Paul M. Frank. Stability analysis and synthesis of nonlinear time-delay systems via linear Takagi–Sugeno fuzzy models. *Fuzzy sets and systems*, 124(2):213–229, 2001.

[18] Talel Zouari, Komi Midzodzi Pekpe, Vincent Cocquempot, and Moufida Ksouri. Active mode recognition of switched nonlinear systems: application to fault detection and isolation. *Asian Journal of Control*, 16(2):345–357, 2014.

[19] Hongyi Li, Chengwei Wu, Shen Yin, and Hak-Keung Lam. Observer-based fuzzy control for nonlinear networked systems under unmeasurable premise variables. *IEEE Transactions on Fuzzy Systems*, 24(5):1233–1245, 2016.

[20] H Ghorbel, A El Hajjaji, M Souissi, and M Chaabane. Robust tracking control for Takagi–Sugeno fuzzy systems with unmeasurable premise variables: Application to tank system. *Journal of Dynamic Systems, Measurement, and Control*, 136(4):041011, 2014.

[21] Tanaka Kazuo, Takayuki Ikeda, and Hua O Wang. Robust stabilization of a class of uncertain nonlinear systems via fuzzy control: Quadratic stabilizability, h_∞ control theory, and linear matrix inequalities. *IEEE Transaction on Fuzzy Systems*, 4(1):1–13, 1996.

[22] Talel Zouari, Komi Midzodzi Pekpe, Vincent Cocquempot, and Moufida Ksouri. Data projection method for sensor fault detection and isolation on nonlinear systems based on Takagi–Sugeno model. In *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, pages 109–114. IEEE, 2011.

[23] Yosr Garbouj, Talel Zouari, and Moufida Ksouri. Faulty mode detection and identification for siso nonlinear switching system. In *2018 5th International Conference on Control, Decision and Information Technologies (CoDIT)*, pages 653–658. IEEE, 2018.

[24] Olivier Bernard and Jean-Luc Gouzé. Closed loop observers bundle for uncertain biotechnological models. *Journal of process control*, 14(7):765–774, 2004.

[25] Alain Rapaport and Denis Dochain. Interval observers for biochemical processes with uncertain kinetics and inputs. *Mathematical biosciences*, 193(2):235–253, 2005.

[26] Damiano Rotondo, Rosa M Fernandez-Canti, Sebastian Tornil-Sin, Joaquim Blesa, and Vicenç Puig. Robust fault diagnosis of proton exchange membrane fuel cells using a Takagi–Sugeno interval observer approach. *International Journal of Hydrogen Energy*, 41(4):2875–2886, 2016.

[27] Denis Efimov, Tarek Raissi, and Ali Zolghadri. Control of nonlinear and LPV systems: interval observer-based framework. *IEEE Transactions on Automatic Control*, 58(3):773–778, 2013.

[28] Thach Ngoc Dinh and Hiroshi Ito. Interval observers for continuous-time bilinear systems with discrete-time outputs. In *Control Conference (ECC), 2016 European*, pages 1418–1423. IEEE, 2016.

[29] Hiroshi Ito and Thach Ngoc Dinh. Interval observers for global feedback control of nonlinear systems with robustness with respect to disturbances. *European Journal of Control*, 39:68–77, 2018.

[30] Rihab Lamouchi, Messaoud Amairi, Tarek Raissi, and Mohamed Aoun. Interval observer design for linear parameter-varying systems subject to component faults. In *Control and Automation (MED), 2016 24th Mediterranean Conference on*, pages 707–712. IEEE, 2016.

[31] Frédéric Mazenc, Thach Ngoc Dinh, and Silviu Iulian Niculescu. Interval observers for discrete-time systems. *International Journal of Robust and Nonlinear Control*, 24(17):2867–2890, 2014.

[32] Frédéric Mazenc and Olivier Bernard. Interval observers for linear time-invariant systems with disturbances. *Automatica*, 47(1):140–147, 2011.

[33] Denis Efimov, Wilfrid Perruquetti, Tarek Raïssi, and Ali Zolghadri. On interval observer design for time-invariant discrete-time systems. In *Control Conference (ECC), 2013 European*, pages 2651–2656. IEEE, 2013.

[34] Tarek Raïssi, Denis Efimov, and Ali Zolghadri. Interval state estimation for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 57(1):260–265, 2012.

[35] Frédéric Mazenc, Thach Ngoc Dinh, and Silviu-Iulian Niculescu. Robust interval observers and stabilization design for discrete-time systems with input and output. *Automatica*, 49(11):3490–3497, 2013.

[36] Thach Ngoc Dinh and Hiroshi Ito. Decentralization of interval observers for robust controlling and monitoring a class of nonlinear systems. *SICE Journal of Control, Measurement, and System Integration*, 10(2):117–123, 2017.

[37] Denis Efimov, Wilfrid Perruquetti, and Jean-Pierre Richard. Interval estimation for uncertain systems with time-varying delays. *International Journal of Control*, 86(10):1777–1787, 2013.

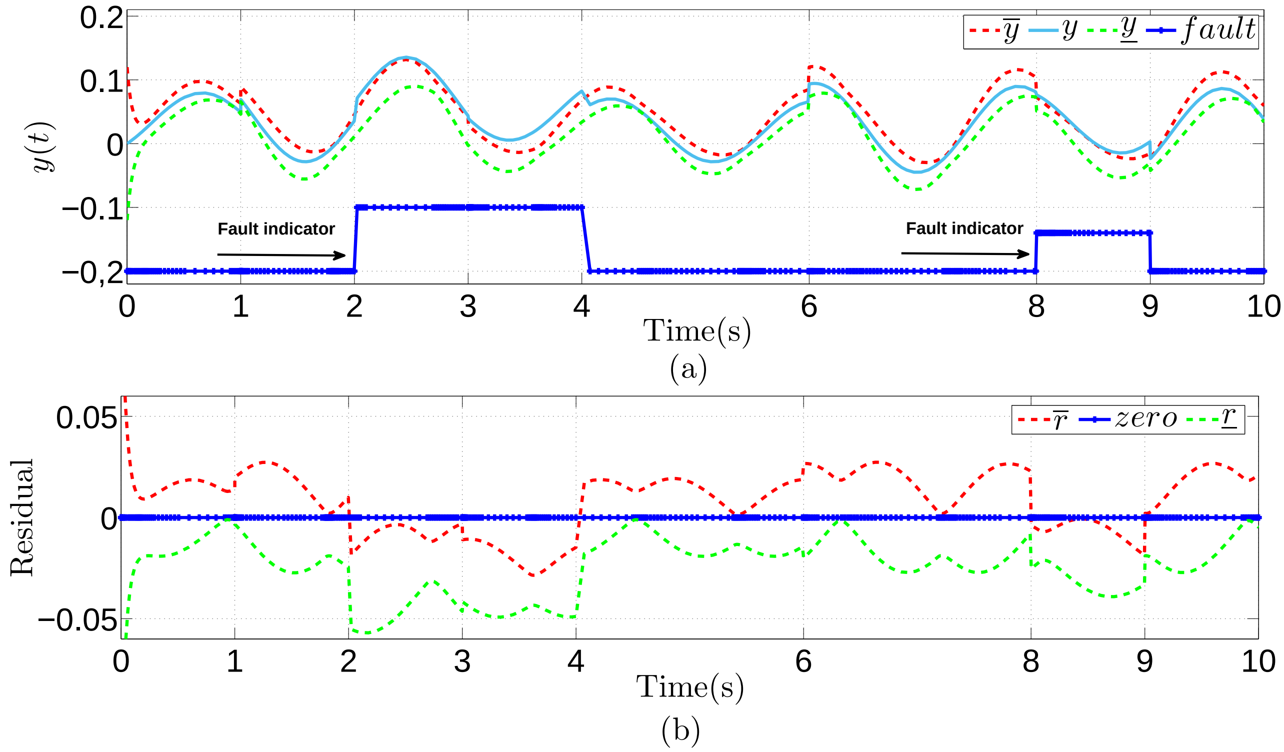

Fig. 5. Interval estimation of (a) the output and (b) the residual

[38] Haifa Ethabet, Djahid Rabehi, Denis Efimov, and Tarek Raïssi. Interval estimation for continuous-time switched linear systems. *Automatica*, 90:230–238, 2018.

[39] Kwassi Holali Degue, Denis Efimov, and Jean-Pierre Richard. Interval observers for linear impulsive systems. In *10th IFAC Symposium on Nonlinear Control Systems (NOLCOS 2016)*, pages 867–872, 2016.

[40] Shenghui Guo and Fanglai Zhu. Interval observer design for discrete-time switched system. In *20th IFAC WC 2017, Toulouse, France*, pages 5073–5078, 2017.

[41] Ghassen Marouani, Thach Ngoc Dinh, Tarek Raïssi, and Hassani Messaoud. Interval observers design for discrete-time linear switched systems. In *16th European Control Conference, Limassol, Cyprus*, pages 801–806, 2018.

[42] Jitao Li, Zhenhua Wang, Yi Shen, and Yan Wang. Interval observer design for discrete-time uncertain Takagi–Sugeno fuzzy systems. *IEEE Transactions on Fuzzy Systems*, 2019.

[43] M Ait Rami, Chun Hung Cheng, and C De Prada. Tight robust interval observers: an lp approach. In *Decision and Control, 2008. CDC 2008. 47th IEEE Conference on*, pages 2967–2972. IEEE, 2008.

[44] Zhenhua Wang, Cheng-Chew Lim, and Yi Shen. Interval observer design for uncertain discrete-time linear systems. *Systems & Control Letters*, 116:41–46, 2018.

[45] Lorenzo Farina and Sergio Rinaldi. *Positive linear systems: theory and applications*, volume 50. John Wiley & Sons, 2011.

[46] Stanislav Chebotarev, Denis Efimov, Tarek Raïssi, and Ali Zolghadri. Interval observers for continuous-time lpv systems with l_1/l_2 performance. *Automatica*, 58:82–89, 2015.

[47] Stephen Boyd, Laurent El Ghaoui, Eric Feron, and Venkataramanan Balakrishnan. *Linear matrix inequalities in system and control theory*, volume 15. Siam, 1994.

[48] Bin Jiang, Jian Liang Wang, and Yeng Chai Soh. An adaptive technique for robust diagnosis of faults with independent effects on system outputs. *International Journal of Control*, 75(11):792–802, 2002.

[49] Linh Vu, Debasish Chatterjee, and Daniel Liberzon. Input-to-state stability of switched systems and switching adaptive control. *Automatica*, 43:639–646, 2007.

[50] Thach Ngoc Dinh, Ghassen Marouani, Tarek Raïssi, Zhenhua Wang, and Hassani Messaoud. Optimal interval observers for discrete-time linear switched systems. *International Journal of Control*, pages 1–9, 2019.

[51] Ester Van Broekhoven and Bernard De Baets. Fast and accurate center of gravity defuzzification of fuzzy system outputs defined on trapezoidal fuzzy partitions. *Fuzzy Sets and Systems*, 157(7):904–918, 2006.

[52] Dalil Ichalal, Benoît Marx, José Ragot, and Didier Maquin. An approach for the state estimation of Takagi–Sugeno models and application to sensor fault diagnosis. In *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, pages 7789–7794. IEEE, 2009.

[53] Johan Lofberg. Yalmip: A toolbox for modeling and optimization in matlab. In *Computer Aided Control Systems Design, 2004 IEEE International Symposium on*, pages 284–289. IEEE, 2004.

[54] Michael Grant, Stephen Boyd, and Yinyu Ye. Cvx: Matlab software for disciplined convex programming (web page and software), http://www.stanford.edu/boyd/cvx/, 2008.

[55] Yongming Li, Xiao Min, and Shaocheng Tong. Adaptive fuzzy inverse optimal control for uncertain strict-feedback nonlinear systems. *IEEE Transaction on Fuzzy Systems, DOI: 10.1109/TFUZZ.2019.2935693*, 2019.

[56] Yongming Li, Kangkang Sun, and Shaocheng Tong. Adaptive fuzzy robust fault-tolerant optimal control for nonlinear large-scale systems. *IEEE Transaction on Fuzzy Systems*, 26(5):2899–2914, 2018.

[57] Hak-Keung Lam. A review on stability analysis of continuous-time fuzzy-model-based control systems: From membership-function-independent tomembership-function-dependent analysis. *Engineering Applications of Artificial Intelligence*, 67:390–408, 2018.