Interval estimation for discrete-time switched linear systems based on L_∞ observer and ellipsoid analysis

Monia Dkhil[1], Thach Ngoc Dinh[2,⋆], Zhenhua Wang[3], Tarek Raïssi[2] and Messaoud Amairi[1]

***Abstract*—This paper proposes a two-step interval estimation method for discrete-time switched linear systems with unknown but bounded uncertainties. Based on the ellipsoidal analysis, the proposed estimator provides upper and lower bounds of the system state with high computational analysis. The size of the obtained ellipsoids is minimized using the trace criterion. The idea of introducing an L_∞ performance is also employed in order to improve the estimation accuracy. Its design conditions are given in terms of Linear Matrix Inequalities (LMIs). Finally, a numerical example emphasizes the effectiveness of the contribution.**

***Index Terms*—Interval estimation, ellipsoid, switched linear systems, L_∞ performance.**

I. Introduction

Over the past decades, state estimation has become one of the critical problems in control theory. For the targets of monitoring, identification and fault detection, the knowledge of all system state variables is a must. However in most of the cases, they may not be completely measurable and moreover, the sensors can be limited in number or provide inaccurate measurements with low reliability. In this framework, observer algorithms were proposed in order to overcome this limitation. They have acquired the growing attention due to their ability to estimate the system state from the input-output process.

Non-exhaustive literature review. As well-known in the literature, dynamical processes are typically affected by known or unknown uncertainties (e.g., noises, disturbances etc.) which are usually assumed to be either stochastic or deterministic. Since 1960s, several results are devoted to the stochastic framework [1], [2]. It concerns the cases of known disturbances and measurement noises that are distinguished by an explicit distribution (generally Gaussian). Nevertheless, it is difficult to check this assumption in real applications. A general and practical assumption is that the uncertainties are unknown but bounded by known bounds [3], [4]. So far, the state estimation issues under this context are fertile ground for studies. Several set-membership estimation techniques which are based for instance on zonotope [5]–[7] or ellipsoid [8], [9] are proposed in order to compute feasible sets that contain all the possible states.

Lately, interval observers have become more popular due to their computational efficiency and their simple principle [10]–[13]. In the literature, it has been shown that the cooperativity assumption is the main design limitation. Therefore, a change of coordinates has been proposed to relax this restrictive requirement [12], [14]. However, state transformation-based design may also cause conservatisms as pointed out in [15]: it is impossible to synthesize simultaneously the framer gain and the coordinate transformation matrix satisfying in the same time the cooperativity constraint and the estimation accuracy. In that context, some recent studies exhibit two-step methods [16]–[18]. The idea in [16], [17] is to combine the observer design with zonotopic techniques, which helps to overcome the cooperative constraint frequently employed in design of interval observers and to improve the accuracy performance of interval estimation. However, the drawback of the zonotope-based estimation is the trade-off between accuracy and computation burdens. During the propagation process, the dimensions of zonotopes grow linearly which increases the computational burden. To handle this limitation, an alternative approach which also provides good performances and relaxes the computational complexity is based on the ellipsoidal representation [19]–[21]. On the one hand, this approach allows getting state bounds in a deterministic and guaranteed way. On the other hand, the simplicity of formulation and its geometry form can be used in order to obtain a more implementable solution. Thus, a good trade-off between computation complexity and estimation accuracy is ensured.

Additionally, switched systems have witnessed an expanding interest [22], [23] due to their specificity to represent several real-life applications such as automatic control systems, network control processes, power electronics processes, and flight control systems [24], [25]. Switched systems have emerged as a particular class of hybrid dynamical systems (HDS). They are composed of a number of continuous or discrete time subsystems called also modes and a switching rule that allows only one mode to be active at each time [25]. Naturally, state interval estimation of such systems is important and has attracted attention [26], [27]. To the best

[1]Monia Dkhil and Messaoud Amairi are with University of Gabes, National Engineering School of Gabes, Electrical Engineering Department, Research laboratory MACS, LR16ES22, Gabes 6029, Tunisia moniadkhil05@gmail.com, amairi.messaoud@ieee.org

[2]Thach Ngoc Dinh and Tarek Raïssi are with Conservatoire National des Arts et Métiers (CNAM), Cedric-Lab, 292 rue St-Martin, 75141 Paris Cedex 03, France ngoc-thach.dinh@lecnam.net, tarek.raissi@cnam.fr

[3]Zhenhua Wang is with School of Astronautics, Harbin Institute of Technology, West Da-Zhi Street, 150001 Harbin, PR China zhenhua.wang@hit.edu.cn

⋆Corresponding author.

of the authors knowledge, two-step interval estimation based on the ellipsoidal approximation for discrete-time switched linear systems has not been investigated. Moreover, notice that observer gains decide also the tightness of the interval width and as proved in [28], the practical signals are not usually energy-bounded but have peak bounds values. Therefore, as a robustness performance, we consider the L_∞ norm to compute these observer gains, which is naturally more consistent with interval analysis since it minimizes the peak-to-peak gain.

Contribution. Based on the arguments given above, this paper proposes a novel state interval estimation method for discrete-time linear switched systems with unknown but bounded uncertainties. The observer design is combined with an ellipsoidal formulation. More precisely, first an L_∞ observer is used to reduce the influence of unknown uncertainties. The observer design conditions can be converted into linear matrix inequalities (LMIs). Next, an ellipsoidal set is computed for the estimation error and generates two bounds for the state. The proposed method provides a systematic way to improve the accuracy of interval estimation. Compared to [29], the contributions and the main difference of this paper is that the changes of coordinates required to ensure the non-negativity of the estimation errors are not needed in the proposed approach. Therefore, the estimation pessimism can be reduced.

The present paper is organized as follows. Some preliminaries are given in Section II. The problem statement is illustrated in Section III. The main results are proved and scheduled in Section IV and V. A numerical example of the proposed observer and some comparison simulations are presented in section VI. Finally, a conclusion is presented in section VII.

II. Preliminaries

Notations

- $\mathbb{R}$ and $\mathbb{N}$ represent the real and natural number sets, respectively.
- $\overline{1,N}$ is the sequence of integers $1,...,N$.
- The identity matrix of any dimension n is denoted by I_n.
- For $x_a = [x_{a,1},...,x_{a,n}]^\top \in \mathbb{R}^n$ and $x_b = [x_{b,1},...,x_{b,n}]^\top \in \mathbb{R}^n$, $x_a \leq x_b$ if and only if, for all $i \in \{1,...,n\}$, $x_{a,i} \leq x_{b,i}$.
- The relation $M \succ 0$ ($M \prec 0$) means that the matrix $M \in \mathbb{R}^{n\times n}$ is positive definite (negative definite).
- For a signal $s_k \in \mathbb{R}^n$, the L_∞ norm is given by $\|s\|_\infty = \sup_{k>0} \|s_k\|$, where $\|s_k\| = \sqrt{{s_k}^T s_k}$.

Definition 1 ([9]). *A non-degenerate ellipsoid set $\xi(x_c, X) \subset \mathbb{R}^n$ is defined by*

$$\xi(x_c, X) \equiv \left\{x : (x - x_c)^T X^{-1}(x - x_c) \leq 1\right\}, \quad (1)$$

where $x_c \in \mathbb{R}^n$ and $X = X^T \succ 0$ are its center and its shape matrix, respectively. □

Definition 2 ([3]). *Let us consider a convex set $\chi \subset \mathbb{R}_n$, its support function in terms of a vector $\ell \in \mathbb{R}_n$ is given as follows*

$$\varphi_\chi(\ell) = \max_{s\in\chi} \ell^T s. \quad (2)$$

□

Property 1 ([9]). *The affine transformation of an ellipsoid $y = Ax + a$ is given as*

$$A\xi(x_c, X) \oplus a = \xi(Ax_c + a, AXA^T), \quad (3)$$

where $A \in \mathbb{R}^{n\times n}$ and $a \in \mathbb{R}^n$. □

Property 2 ([9]). *The Minkowski sum of two ellipsoids $\xi_1(x_{c1}, X_1)$ and $\xi_2(x_{c2}, X_2)$ satisfies*

$$\xi_1(x_{c1}, X_1) \oplus \xi_2(x_{c2}, X_2) \subseteq \xi_3(x_{c3}, X_3), \quad (4)$$

where $x_{c3} = x_{c1} + x_{c2},\ X_3 = \varphi_1^{-1}X_1 + \varphi_2^{-1}X_2$, such that $\varphi_1 + \varphi_2 = 1$, $\varphi_1 > 0$ and $\varphi_2 > 0$. □

Lemma 1 ([3]). *Given a non-degenerate ellipsoid $\xi(x_c, X)$ defined as* (1)*, its support function in terms of $\ell \in \mathbb{R}_n$ is*

$$\varphi_{\xi(x_c,X)}(\ell) = \ell^T x_c + \sqrt{\ell^T X \ell}. \quad (5)$$

□

III. Problem Statement

In the present paper we consider the following discrete-time switched linear system

$$\begin{cases} x_{k+1} = A_q x_k + B_q u_k + F_q w_k \\ y_k = C_q x_k \end{cases}, \forall q \in \overline{1,N},\ N \in \mathbb{N}, \quad (6)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $w \in \mathbb{R}^n$ are the state, the input, the output and the unknown disturbance vectors, respectively. The invariant-time matrices A_q, B_q, F_q and C_q have appropriated dimensions. N defines the number of subsystems and q is the index of the active mode at each time k. For the rest of paper, we need the following assumptions.

Assumption 1. The pair (A_q, C_q) is detectable.

Assumption 2. The disturbance w_k is bounded and belongs into an appropriate ellipsoidal set defined as follows

$$\xi(0, W) \equiv \left\{w : w^T W^{-1} w \leq 1\right\}, \quad (7)$$

where W is a positive definite matrix which represents the shape of this ellipsoid.

Assumption 3. The initial system state is supposed to be unknown but included in the following ellipsoid set

$$\xi(x_{c0}, X_0) \equiv \left\{x : (x - x_{c0})^T X_0^{-1}(x - x_{c0}) \leq 1\right\}. \quad (8)$$

Where $X_0 = X_0^T \succ 0$.

In the sequel, the aim is to design a two-step interval estimation through the combination of an L_∞ observer performance with the ellipsoidal approach to provide two signals $\underline{x}_k$ and $\overline{x}_k$ such that

$$\underline{x}_k < x_k < \overline{x}_k, \forall k > 0. \quad (9)$$

The L_∞ formalism is used to attenuate the system disturbances effects and to improve the estimation accuracy.

IV. L_∞ FORMALISM FOR SWITCHED LINEAR SYSTEMS

Consider the following Luenberger observer structure for the switched system (6)

$$\hat{x}_{k+1} = (A_q - L_qC_q)\hat{x}_k + B_qu_k + L_qy_k. \quad (10)$$

Where $\hat{x}_k \in \mathbb{R}^n$ is the state estimation vector and $L_q \in \mathbb{R}^{n\times n}$ is the observer gain matrix.

At a given instant k, the estimation error is defined by

$$e_k = x_k - \hat{x}_k. \quad (11)$$

Thus,

$$e_{k+1} = \overline{A}_q e_k + F_q w_k, \quad (12)$$

where $\overline{A}_q = (A_q - L_qC_q)$.

The aim is to design the observer (10) satisfying

- The nominal system of (12) is stable.
- e_k should satisfy the following L_∞ performance, i.e.,

$$\|e_k\| \leq \sqrt{\gamma_w(\lambda(1-\lambda)^k V_0 + \gamma_w \widetilde{w}^2)}, \quad (13)$$

where $P \succ 0 \in \mathbb{R}^{n\times n}$, $V_0 = e_0^T P e_0$, $\gamma_w > 0$, and $0 < \lambda < 1$.

The following theorem is proposed to design the gain matrix in observer (10).

Theorem 1. *Let Assumptions 1, 2 and 3 hold and given parameters $\gamma_w > 0$, and $0 < \lambda < 1$. If there exist positive definite matrices $P \in \mathbb{R}^{n\times n}$, $U_q \in \mathbb{R}^{n\times p}$, and a constant $\sigma > 0$, $\forall q \in \overline{1, N}$ such that*

$$\begin{bmatrix} (\lambda-1)P & 0 & (PA_q - W_qC_q)^T \\ 0 & -\sigma I_n & (PF_q)^T \\ (PA_q - W_qC_q) & (PF_q) & -P \end{bmatrix} \prec 0, \quad (14)$$

$$\begin{bmatrix} \lambda P & 0 & I_n \\ 0 & (\gamma_w - \sigma)I_n & 0 \\ I_n & 0 & \gamma_w I_n \end{bmatrix} \succ 0. \quad (15)$$

Then, the dynamics (12) are input-to-state stable and the observer gain is chosen as $L_q = P^{-1}W_q$. □

Remark 1. Note that the parameter γ_w can be optimized.

Proof 1. *Let us consider the following Lyapunov function for (12)*

$$V_k = e_k^T P e_k, P \succ 0,$$

that can be shown to satisfy

$$\triangle V_K = V_{k+1} - V_k < 0,$$

when $w_k = 0$. Indeed, from (12) we have

$$e_{k+1}^T P e_{k+1} - e_k^T P e_k = (\overline{A}_q e_k + F_q w_k)^T P (\overline{A}_q e_k + F_q w_k) - e_k^T P e_k.$$

Thus,

$$\triangle V_k = \begin{bmatrix} e_k \\ w_k \end{bmatrix}^T \begin{bmatrix} \overline{A}_q{}^T P \overline{A}_q - P & \overline{A}_q{}^T P F_q \\ F_q^T P \overline{A}_q & F_q^T P F_q \end{bmatrix} \begin{bmatrix} e_k \\ w_k \end{bmatrix}.$$

Now, bearing in mind that when (14) is accomplished, by substituting $W_q = PL_q$ and according to $\overline{A}_q = (A_q - L_qC_q)$, the inequality (14) can be rewritten as follows

$$\begin{bmatrix} (\lambda-1)P & 0 & (P\overline{A}_q)^T \\ 0 & -\sigma I_n & (PF_q)^T \\ (P\overline{A}_q) & (PF_q) & -P \end{bmatrix} \prec 0. \quad (16)$$

Now, let us consider the following variable

$$\Psi = \begin{bmatrix} I_n P & 0 & \overline{A}_q^T \\ 0 & I_n & F_q^T \end{bmatrix}. \quad (17)$$

Then, by pre- and post-multiplying (16) with (17) and its transpose, respectively, we obtain

$$\begin{bmatrix} (\lambda-1)P + \overline{A}_q^T P \overline{A}_q & \overline{A}_q^T P F_q \\ F_q^T P \overline{A}_q & \sigma I_n + F_q^T P F_q \end{bmatrix} \prec 0, \quad (18)$$

which is equivalent to

$$\begin{bmatrix} \overline{A}_q{}^T P \overline{A}_q - P & \overline{A}_q{}^T P F_q \\ F_q^T P \overline{A}_q & F_q^T P F_q \end{bmatrix} + \begin{bmatrix} \lambda P & 0 \\ 0 & -\sigma I_n \end{bmatrix} \prec 0. \quad (19)$$

Pre- and post-multiplying (19) with $\begin{bmatrix} e_k^T & w_k^T \end{bmatrix}$ and its transpose, respectively, we acquire

$$\triangle V_k + \lambda V_k - \sigma w_k^T w_k \prec 0. \quad (20)$$

When $w_k = 0$, the inequality (20) implies that

$$\triangle V_k = V_{k+1} - V_k < -\lambda V_k < 0. \quad (21)$$

Thus, the error dynamic (12) is input-to state stable.

Next, it remains only to verify that

$$\|e_k\| \leq \sqrt{\gamma_w(\lambda(1-\lambda)^k V_0 + \gamma_w \widetilde{w}^2)}. \quad (22)$$

From inequality (20), we have

$$V_{k+1} < (1-\lambda)V_k + \sigma \widetilde{w}^2, \quad (23)$$

which implies

$$\begin{aligned} V_{k+1} <& (1-\lambda)^k V_0 + \sigma \sum_{\tau=0}^{k-1} (1-\lambda)^\tau \widetilde{w}^2 \\ \leq& (1-\lambda)^k V_0 + \sigma \frac{1-\lambda^k}{\lambda} \widetilde{w}^2 \\ \leq& (1-\lambda)^k V_0 + \sigma \frac{\widetilde{w}^2}{\lambda}. \end{aligned} \quad (24)$$

Using the Schur complement, the inequality (15) is equivalent to

$$\begin{bmatrix} \lambda P & 0 \\ 0 & (\gamma_w - \sigma)I_n \end{bmatrix} - \frac{1}{\gamma_w} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \end{bmatrix} \succ 0. \quad (25)$$

Now, by pre-and post-multiplying (25) with $\begin{bmatrix} e_k^T & w_k^T \end{bmatrix}$ and its transpose, respectively, we obtain

$$e_k^T e_k \leq \gamma_w(\lambda V_k + (\gamma_w - \sigma)\widetilde{w}^2). \quad (26)$$

Then by setting (24) into (27), we have

$$\begin{aligned} e_k^T e_k \leq& \gamma_w(\lambda((1-\lambda)^k V_0 + \sigma \frac{\widetilde{w}^2}{\lambda} + (\gamma_w - \sigma)\widetilde{w}^2) \\ \leq& \gamma_w(\lambda(1-\lambda)^k V_0 + \gamma_w \widetilde{w}^2). \end{aligned} \quad (27)$$

Then, the condition in (22) is satisfied. □

V. Ellipsoidal State Bounding Process For Discrete-Time Switched Linear System

In this section, the aim is to find a feasible set $\xi(x_{ck}, X_k)$ which contains the system state x_k, in order to derive the two signals $\overline{x}_k$ and $\underline{x}_k$ satisfying (9). Theorems 2 and 3 are given for such purpose.

Theorem 2. *Consider the observer* (10) *for the system (6) and assume that Assumption 3 is satisfied, with $x_{c0} = \hat{x}_0$. The state x_k belongs into a feasible ellipsoidal set $\xi(\hat{x}_k, X_k)$, such that*

$$X_{k+1}(\delta) = (1 + \frac{1}{\delta})\overline{A}_q X_k \overline{A}_q{}^T + (1+\delta)F_q W_k F_q{}^T, \quad (28)$$

with

$$\delta = \sqrt{\frac{trace(\overline{A}_q X_k \overline{A}_q^T)}{trace(F_q W F_q^T)}}. \quad (29)$$

Proof. From (11) and by considering $x_{c0} = \hat{x}_0$, we verify that

$$e_0 \in \xi(x_{c0}, X_0) \oplus (-\hat{x}_0) = \xi(0, X_0). \quad (30)$$

Based on *Assumption 2* and (30), we have $e_k \in \xi(0, X_k)$. Thus the system state satisfies

$$x_k \in \xi(0, X_k) \oplus (\hat{x}_k) = \xi(\hat{x}_k, X_k). \quad (31)$$

Now, according to the previous assumptions, the observer system (10) and using the ellipsoidal analysis procedure, we have

$$e_{k+1} \in \overline{A}_q \xi(0, X_k) \oplus F_q \xi(0, W_k) = \xi(0, X_{k+1}(\delta)). \quad (32)$$

Applying Property 1, we obtain

$$e_{k+1} \in \xi(0, \overline{A}_q X_k \overline{A}_q{}^T) \oplus \xi(0, F_q W_k F_q^T). \quad (33)$$

Thus

$$e_{k+1} \in \xi(0, X_{k+1}(\delta)), \quad (34)$$

which yields

$$x_{k+1} = \hat{x}_{k+1} + e_{k+1} \in \xi(\hat{x}_{k+1}, X_{k+1}(\delta)). \quad (35)$$

Therefore, we conclude that

$$x_k \in \xi(\hat{x}_k, X_k),\ \forall k \geq 0. \quad (36)$$

Now, according to the Property 2, the shape matrix $X_{k+1}(\delta)$ is defined as follows

$$X_{k+1}(\delta) = (1 + \frac{1}{\delta})\overline{A}_q X_k \overline{A}_q{}^T + (1+\delta)F_q W_k F_q{}^T, \quad (37)$$

where δ minimizes the size of the feasible ellipsoidal set at time $k+1$ using the trace criterion.

Remark 2. Note that the sum of two ellipsoids is not an ellipsoid and the outer approximation is optimized in the following by using the trace criterion.

Trace criterion

Let us consider the following trace function

$$\begin{aligned} tr_k &= trace(X_{k+1}) \quad (38) \\ &= trace((1 + \frac{1}{\delta})\overline{A}_q X_k \overline{A}_q{}^T + (1+\delta)F_q W_k F_q{}^T). \end{aligned}$$

Due to the convexity of the trace function [9, Lemma 3.1] with respect to δ, the minimum value of δ based on the trace criterion is given by

$$\begin{aligned} \frac{\partial tr_k}{\partial \delta} &= -\delta^{-2} trace(\overline{A}_q X_k \overline{A}_q{}^T) + trace(F_q W F_q^T) \quad (39) \\ &= 0. \end{aligned}$$

Hence, an explicit solution of δ can be deduced as

$$\delta = \sqrt{\frac{trace(\overline{A}_q X_k \overline{A}_q^T)}{trace(F_q W F_q^T)}}. \quad (40)$$

□

Now, after obtaining the feasible set $\xi(\hat{x}_k, X_k)$, the signals $\overline{x}_k$ and $\underline{x}_k$ are computed using the following theorem.

Theorem 3. *For the system* (6)*, the two bounds $\underline{x}_k$ and $\overline{x}_k$ satisfying* (9) *can be obtained as follows*

$$\begin{cases} \overline{x}_k(i) = \hat{x}_{c_k}(i) + \overline{e}_k(i),\ i = 1..n \\ \underline{x}_k(i) = \hat{x}_{c_k}(i) + \underline{e}_k(i),\ i = 1..n \end{cases} \quad (41)$$

where $\underline{e}_k = -\overline{e}_k$ and

$$\overline{e}_k(i) = \sqrt{X_k(i,i)} \quad (42)$$

Proof. From Theorem 3, we have verified that $x_k \in \xi(\hat{x}_k, X_k), \forall k \geq 0$ and $e_k \in \xi(0, X_k)$. By defining $\ell_i \in \mathbb{R}_n$ as the vector with its $i-$th element equal to 1 and the others to 0 and according to Lemma 1, we have

$$\begin{aligned} \varphi_{\xi(0,X)}(\ell_i) &= \sqrt{\ell_i^T X \ell_i} = \sqrt{\ell^T(i) X \ell(i)} \quad (43) \\ &= \sqrt{X(i,i)}. \quad (44) \end{aligned}$$

According to Definition 2, we have

$$\varphi_{\xi(0,X)}(\ell_i) = \max_{e_k \in \xi(0,X)} \ell_i^T e_k = \overline{e}_k(i). \quad (45)$$

Thus, (46) and (43) imply that

$$\varphi_{\xi(0,X)}(\ell_i) = \sqrt{X(i,i)} = \overline{e}_k(i). \quad (46)$$

Then, from Definition 2, we have $e_k(i) = \ell_i^T e_k \leq \varphi_{\xi_k(0,X_k)}(\ell_i),\ \forall i = \overline{1,n}$.
Thus, $e_k \leq \overline{e}_k$, then since $\xi_k(0, X_k)$ is symmetric, involving that $-e_k \in \xi_k(0, X_k)$. As well, from Definition 2, we obtain $-e_k \leq \overline{e}_k$ then $e_k \geq -\overline{e}_k = \underline{e}_k$. Therefore, we can conclude that $\hat{x}_k + \underline{e}_k \leq x_k \leq \hat{x}_k + \overline{e}_k$. □

VI. Simulation results

This section presents some simulation results of the proposed method. In order to substantiate the efficiency of the suggested approach, the results of this paper are compared with [29]. Let us consider the switched linear system (6) with three modes ($N = 3$), where

$$A_1 = \begin{pmatrix} 0.53 & 0.01 & 0.12 \\ 0.32 & -0.08 & 0.15 \\ 0.47 & 0.13 & 0.54 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0.3 & 0.05 & 0.43 \\ 0.35 & 0.12 & 0.43 \\ 0.5 & 0.25 & 0.23 \end{pmatrix}, A_3 = \begin{pmatrix} 0.4 & 0.31 & 0.28 \\ 0.1 & 0.3 & 0.2 \\ 1 & 0.5 & 0.04 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0.4 & 1.2 & 0.5 \end{pmatrix}, C_2 = \begin{pmatrix} 0.71 & 0.81 & 0.6 \end{pmatrix},$$
$$C_3 = \begin{pmatrix} 0.85 & 0.3 & 0.7 \end{pmatrix},$$
$$B_1 = \begin{pmatrix} 1.5 \\ 1 \\ 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 \\ 1 \\ 0.4 \end{pmatrix}, B_3 = \begin{pmatrix} 0.4 \\ 1 \\ 1 \end{pmatrix},$$

$F_q = I_3$, $\forall q \in \overline{1,3}$, $w_k = 0.1 \begin{pmatrix} \sin(0.5k) \\ \cos(0.5k) \\ \cos(0.5k) \end{pmatrix}$ is the bounded state disturbances with $\bar{w} = \begin{pmatrix} 0.1 \\ 0.1 \\ 0.1 \end{pmatrix}$ such that $w_k^T w_k \leq \bar{w}^T \bar{w}$.

According to the proposed optimisation algorithm, we choose $\lambda = 0.2$. It is worth noting that λ is a tuning parameter which we employ heuristics to select. The feasible solutions of the LMIs (14) and (15) are obtained as follows $\sigma = 3.1886$, $\gamma_w = 3.1886$, and

$$L_1 = \begin{pmatrix} 0.2233 \\ 0.1064 \\ 0.4407 \end{pmatrix}, L_2 = \begin{pmatrix} 0.3635 \\ 0.4208 \\ 0.4646 \end{pmatrix},$$
$$L_3 = \begin{pmatrix} 0.4870 \\ 0.2598 \\ 0.7460 \end{pmatrix}.$$

The existence of such feasible solutions of (14) and (15) implies that the error dynamic (12) is input-to-state stable and e_k satisfies the L_∞ performance (13).

Now, applying the method proposed in [29, Theorem 3] to the simulation system (6) yields

$$L_1 = \begin{pmatrix} 0.3387 \\ 0.2476 \\ 0.7304 \end{pmatrix}, L_2 = \begin{pmatrix} 0.4842 \\ 0.5236 \\ 0.4547 \end{pmatrix},$$
$$L_3 = \begin{pmatrix} 0.4782 \\ 0.2912 \\ 0.5695 \end{pmatrix}.$$

Note that the matrices $A_q - L_q C_q$ are not nonnegative for all $q \in \overline{1,3}$. Then, according to the approach in [29], changes of coordinates $z = R_q x$ are required such that $R_q(A_q - L_q C_q)R_q^{-1}$ are nonnegative. The matrices R_q can be computed as follows

$$R_1 = \begin{pmatrix} -0.3931 & 0.0408 & 0.1720 \\ -2.8860 & 7.4643 & -1.2225 \\ 3.2791 & -75051 & 2.0504 \end{pmatrix},$$
$$R_2 = \begin{pmatrix} 5.7701 & -6.7259 & 0.6 \\ -0.0653 & -0.1854 & 0.0953 \\ -5.7048 & 6.9113 & 0.3047 \end{pmatrix},$$
$$R_3 = \begin{pmatrix} -3.8872 & 4.0362 & 0.4411 \\ -3.5690 & 0.1117 & 1.6520 \\ 7.4562 & -4.1479 & -1.0931 \end{pmatrix}.$$

In the simulation, the initial state is $x_0 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$, the initial observer state $\hat{x}_0 = \begin{pmatrix} 0.5 & 0.5 & 0.5 \end{pmatrix}^T$. The initial state ellipsoid set is defined by: its center $c_0 = (0 \quad 0 \quad 0)^T$, and its shape matrix $X_0 = 10 I_3$. The disturbances ellipsoid set is defined by its shape matrix $W = \begin{pmatrix} 0.23 & 0.022 & 0.022 \\ 0.022 & 0.23 & 0.022 \\ 0.022 & 0.022 & 0.23 \end{pmatrix}$.

The switching signal, which governs the switching between the subsystems, is plotted Figure 1. Figure 2 presents the state x, the feasible ellipsoid set $\xi(\hat{x}_k, X_k)$ for $k = 0$, $k = 50$, the disturbance w_k and the ellipsoid set uncertainties $\xi(0, W)$. It shows that the proposed approach allows getting at each instant a feasible set $\xi(\hat{x}_k, X_k)$ containing the system state x_k. It is also worth noting that the disturbance w_k belongs into the ellipsoidal set uncertainties $\xi(0, W)$ so Assumption 2 is satisfied.

The simulation results of the proposed interval estimator as well as the comparison with [29] for both coordinates are illustrated in Figure 3 where solid lines refer to the system state and the method in [29] while dashed lines refer to the proposed approach in this paper. The approach proposed in [29] suffers from using the change of coordinates which causes pessimism while the two-step interval estimation proposed in the present paper integrates robust observer design with reachability analysis of the error dynamics and gets rid of the cooperativity requirement. Therefore, the estimation pessimism can be reduced. This comparative analysis is illustrated by Figure 3.

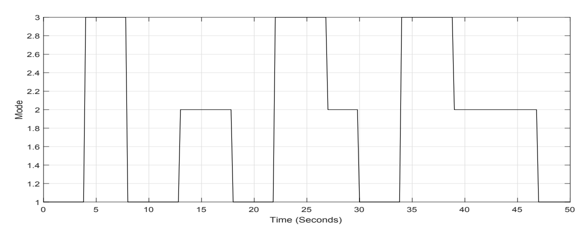

Fig. 1. Switching signal.

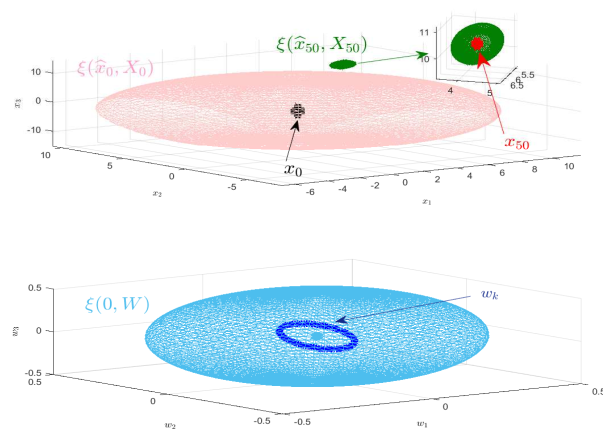

Fig. 2. System state x, ellipsoid sets $\xi(\hat{x}_k, X_k)$ at instant $k = 0$, $k = 50$, w_k and $\xi(0, W)$.

Figure 3 shows that despite uncertainties, for both above-mentioned approaches, the system state is always framed by the upper and the lower estimates which point out that the relation $\underline{x}_k \leq x_k \leq \overline{x}_k$ is satisfied during the whole estimation process. Furthermore, with the same switched system, according to Figure 3, we observe that the interval width obtained by our approach is more narrow than the one proposed by [29]. The tighter the interval width is, the more the estimated bounds $\overline{x}_k$ and $\underline{x}_k$ approach to the real system state x_k. Thus, the estimation accuracy is improved.

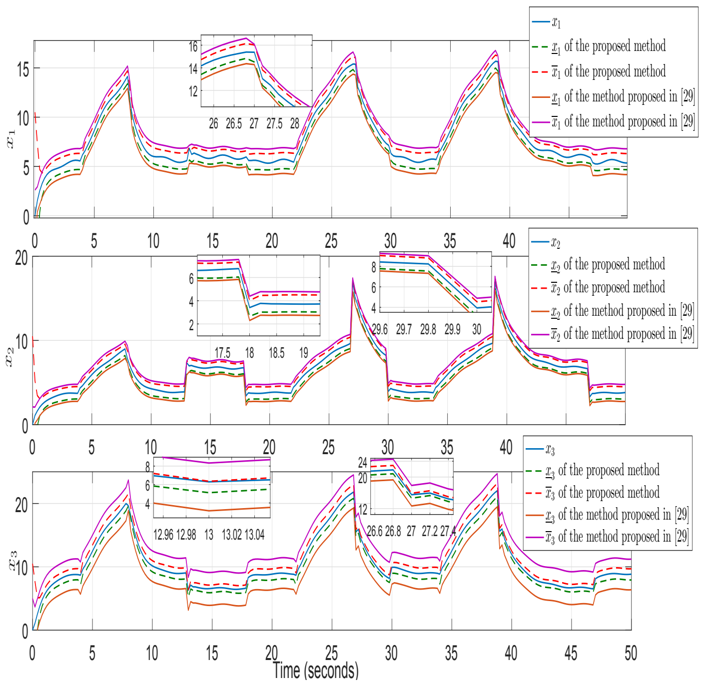

Fig. 3. Comparative simulation with [29].

VII. Conclusion

State estimation of discrete-time switched linear system subject to unknown but bounded uncertainties has been considered in this paper. A new methodology to design a two-step interval estimation based on the ellipsoidal approximation which allows one to derive two signals $\underline{x}_k$ and $\overline{x}_k$ such that $\underline{x}_k < x_k < \overline{x}_k$ for all $k \geq 0$ has been investigated. For the reason of estimation performance, the present approach has also integrated an L_∞ formalism to improve the estimation accuracy by attenuating the system disturbances effects. A comparative simulation shows the efficiency of the proposed contribution. For a further work, this methodology will be extended to deal with switched systems with unknown switching rules.

References

[1] N. A. Nguyen, "Stochastic output feedback control: Convex lifting approach," *Automatica*, vol. 89, pp. 212–220, 2018.

[2] H. Sorenson, "Special issue on applications of kalman filtering," *IEEE Transactions on Automatic Control*, vol. 28, no. 3, pp. 253–434, 1983.

[3] F. Schweppe, "Recursive state estimation: Unknown but bounded errors and system inputs," *IEEE Transactions on Automatic Control*, vol. 13, no. 1, pp. 22–28, 1968.

[4] H. Witsenhausen, "Sets of possible states of linear systems given perturbed observations," *IEEE Transactions on Automatic Control*, vol. 13, no. 5, pp. 556–558, 1968.

[5] E. Fogel and Y.-F. Huang, "On the value of information in system identification-bounded noise case," *Automatica*, vol. 18, no. 2, pp. 229–238, 1982.

[6] D. Bertsekas and I. Rhodes, "Recursive state estimation for a set-membership description of uncertainty," *IEEE Transactions on Automatic Control*, vol. 16, no. 2, pp. 117–128, 1971.

[7] C. Combastel, "Merging kalman filtering and zonotopic state bounding for robust fault detection under noisy environment," *IFAC-PapersOnLine*, vol. 48, no. 21, pp. 289–295, 2015.

[8] S. B. Chabane, C. S. Maniu, T. Alamo, E. F. Camacho, and D. Dumur, "A new approach for guaranteed ellipsoidal state estimation," *IFAC Proceedings Volumes*, vol. 47, no. 3, pp. 6533–6538, 2014.

[9] C. Durieu, E. Walter, and B. Polyak, "Multi-input multi-output ellipsoidal state bounding," *Journal of Optimization Theory and Applications*, vol. 111, no. 2, pp. 273–303, 2001.

[10] J. Li, Z. Wang, W. Zhang, T. Raïssi, and Y. Shen, "Interval observer design for continuous-time linear parameter-varying systems," *Systems & Control Letters*, vol. 134, 2019, https://doi.org/10.1016/j.sysconle.2019.104541.

[11] F. Mazenc, T. N. Dinh, and S.-I. Niculescu, "Interval observers for discrete-time systems," *International Journal of Robust and Nonlinear Control*, vol. 24, pp. 2867–2890, 2014.

[12] T. Raïssi, D. Efimov, and A. Zolghadri, "Interval state estimation for a class of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 1, pp. 260–265, 2012.

[13] Z. Wang, C.-C. Lim, and Y. Shen, "Interval observer design for uncertain discrete-time linear systems," *Systems & Control Letters*, vol. 116, pp. 41–46, 2018.

[14] F. Mazenc and O. Bernard, "Interval observers for linear time-invariant systems with disturbancess," *Automatica*, vol. 47, pp. 140–147, 2011.

[15] E. Chambon, L. Burlion, and P. Apkarian, "Overview of LTI interval observer design: towards a non-smooth optimisation-based approach," *IET Control Theory and Applications*, vol. 10, no. 11, pp. 1258–1268, 2016.

[16] W. Tang, Z. Wang, Q. Zhang, and Y. Shen, "Set-membership estimation for linear time-varying descriptor systems," *Automatica*, vol. 115, 2020, https://doi.org/10.1016/j.automatica.2020.108867.

[17] W. Tang, Z. Wang, Y. Wang, T. Raïssi, and Y. Shen, "Interval estimation methods for discrete-time linear time-invariant systems," *IEEE Transactions on Automatic Control*, vol. 64, no. 11, pp. 4717–4724, 2019.

[18] T. Alamo, J. M. Bravo, and E. F. Camacho, "Guaranteed state estimation by zonotopes," *Automatica*, vol. 41, no. 6, pp. 1035–1043, 2005.

[19] A. A. Kurzhanskiy and P. Varaiya, "Ellipsoidal techniques for reachability analysis of discrete-time linear systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 1, pp. 26–38, 2007.

[20] A. B. Kurzhanski and P. Varaiya, "Ellipsoidal techniques for reachability analysis: internal approximation," *Systems & Control letters*, vol. 41, no. 3, pp. 201–211, 2000.

[21] L. Asselborn, D. Gross, and O. Stursberg, "Control of uncertain nonlinear systems using ellipsoidal reachability calculus," *IFAC Proceedings Volumes*, vol. 46, no. 23, pp. 50–55, 2013.

[22] W. Xu, Z. G. Feng, G.-H. Lin, K. F. C. Yiu, and L. Yu, "Optimal switching of switched systems with time delay in discrete time," *Automatica*, vol. 112, p. 108696, 2020.

[23] X. Xiao, J. H. Park, L. Zhou, and G. Lu, "Event-triggered control of discrete-time switched linear systems with network transmission delays," *Automatica*, vol. 111, p. 108585, 2020.

[24] M. S. Branicky, "Multiple lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 475–482, 1998.

[25] H. Lin and P. J. Antsaklis, "Stability and stabilizability of switched linear systems: a survey of recent results," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 308–322, 2009.

[26] H. Ethabet, T. Raïssi, M. Amairi, and M. Aoun, "Interval observers design for continuous-time linear switched systems," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 6259–6264, 2017.

[27] H. Ethabet, D. Rabehi, D. Efimov, and T. Raïssi, "Interval estimation for continuous-time switched linear systems," *Automatica*, vol. 90, pp. 230–238, 2018.

[28] M. Zhou, M. Rodrigues, Z. Wang, Y. Shen, and D. Theilliol, "H_-/L_∞ fault detection observer for discrete-time Takagi-Sugeno fuzzy systems," in *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*. IEEE, 2017, pp. 5137–5142.

[29] T. N. Dinh, G. Marouani, T. Raïssi, Z. Wang, and H. Messaoud, "Optimal interval observers for discrete-time linear switched systems," *International Journal of Control*, pp. 1–9, 2019.