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ON UNBOUNDED OPTIMAL CONTROLS IN COEFFICIENTS FOR ILL-POSED ELLIPTIC DIRICHLET BOUNDARY VALUE PROBLEMS

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We consider an optimal control problem associated to Dirichlet boundary value problem for linear elliptic equations on a bounded domain Ω. We take the matrix-valued coefficients $A(x)$ of such system as a control in $L^1(\Omega;\mathbb{R}^N\times\mathbb{R}^N)$. One of the important features of the admissible controls is the fact that the coefficient matrices $A(x)$ are non-symmetric, unbounded on Ω, and eigenvalues of the symmetric part $A^{sym}=(A+A^t)/2$ may vanish in Ω.

Key words: degenerate elliptic equations, control in coefficients, weighted Sobolev spaces, variational convergence.

1. Introduction

The aim of this paper is to study the following optimal control problem (OCP) for a linear elliptic equation with unbounded coefficients in the main part of the elliptic operator

$$\left\{\begin{array}{c}\text{Minimize } I(A,y)=\|y-y_d\|^2_{L^2(\Omega)}+\displaystyle\int_\Omega(\nabla y,A^{sym}\nabla y)_{\mathbb{R}^N}\,dx\\ \text{subject to the constraints}\\ -\mathrm{div}\,\big(A(x)\nabla y\big)+a_0(x)y=-\mathrm{div}\,f\quad\text{in}\ \ \Omega,,\\ y=0\quad\ \text{on}\ \partial\Omega,\end{array}\right. \tag{1.1}$$

where the matrix $A=A^{sym}+A^{skew}\in L^1(\Omega;\mathbb{S}^N_{sym})\oplus L^{2p}(\Omega;\mathbb{S}^N_{skew})$ is adopted as a control, $f\in\mathcal{D}'(\Omega;\mathbb{R}^N)$ and $y_d\in L^2(\Omega)$ are given distributions, $p\ge 1$, and $a_0\in L^\infty(\Omega)$ is such that $a_0(x)\ge\alpha>0$ almost everywhere in Ω. We define a class of admissible controls $\mathfrak{A}_{ad}$ as a nonempty compact subset of $L^1(\Omega;\mathbb{S}^N_{sym})\oplus L^{2p}(\Omega;\mathbb{S}^N_{skew})$ such that for every $A\in\mathfrak{A}_{ad}$ we have

$$A^*(x)\preceq A^{skew}(x)\preceq A^{**}(x)\ \text{a.e. in}\ \ \Omega,$$

$$\zeta_{ad}(x)I\le A^{sym}(x)\le\beta(x)I\quad\text{a. e. in}\ \ \Omega,\qquad\int_\Omega A^{sym}(x)\,dx=M,$$

where $M \in \mathbb{S}^N_{sym}$ and $A^*, A^{**} \in L^{2p}(\Omega; \mathbb{S}^N_{skew})$ are given nonzero matrices, $\beta \in L^1(\Omega)$, $\beta \geq \zeta_{ad}$, and $\zeta_{ad}^{-1} \in L^{2q}(\Omega)$ for $q = p/(p-1)$.

This kind of problems naturally appears in the optimal design theory for linearized elliptic boundary value problems. Their characteristic feature of the problem (1.1) is the fact that the existence, uniqueness, and variational properties of the weak solution to (1.1) are drastically different from the corresponding properties of solutions to the elliptic equations with coercive L^∞-matrices in coefficients. Typically, in such cases, the boundary value problem (1.1) with unbounded matrices $A \in \mathfrak{A}_{ad}$ may admit many or even infinitely many weak solutions [21, 22].

Optimal control in coefficients for partial differential equations is a classical subject initiated by Lurie [17], Lions [16], and Pironneau [19]. Since the range of such optimal control problems is very wide, including as well optimal shape design problems, some problems originating in mechanics and others, this topic has been widely studied by many authors. However, most of these results and methods rely on linear PDEs with bounded coefficients in the main part of elliptic operators, while only a few articles deal with with unbounded and degenerate coefficients, see [1, 3, 7–11, 13, 14].

The principal feature of OCP (1.1) is that the corresponding boundary value problem $(1.1)_2$–$(1.1)_3$ is ill-possed and the class of admissible controls $A \in \mathfrak{A}_{ad}$ belongs to $L^1(\Omega; \mathbb{M}^N)$. We note that these assumptions on the class of admissible controls together with L^{2p}-properties of the skew-symmetric parts are essentially weaker than they usually are in the literature. In Sections 2 and 3, we discuss some auxiliary results that are closely related with the correctness of the notion of weak solutions to the above boundary value problem and describe a mathematical background for convergence formalism in variable Sobolev spaces.

We give the precise definition of the class of admissible controls in Section 4 and, using the direct method in the Calculus of variations, we show that a set of optimal pairs to the above problem is nonempty provided the so-called non-triviality condition on the set of admissible solutions. Since this condition is closely related with the existence of weak solutions to the boundary value problem $(1.1)_2$–$(1.1)_3$, we show in Section 5 that this question can be solved due to the approximation approach.

2. Notation and Preliminaries

Let Ω be a bounded open connected subset of $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz boundary $\partial\Omega$. Let χ_E be the characteristic function of a subset $E \subset \Omega$, i.e. $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ if $x \notin E$.

Let $\mathbb{M}^N$ be the set of all $N \times N$ real matrices. We denote by $\mathbb{S}^N_{skew}$ the set of all skew-symmetric matrices $C = [c_{ij}]_{i,j=1}^N$, i.e., C is a square matrix whose transpose is also its opposite. Thus, if $C \in \mathbb{S}^N_{skew}$ then $c_{ij} = -c_{ji}$ and, hence, $c_{ii} = 0$. Therefore, the set $\mathbb{S}^N_{skew}$ can be identified with the Euclidean space $\mathbb{R}^{\frac{N(N-1)}{2}}$. Let

$\mathbb{S}^N_{sym}$ be the set of all $N \times N$ symmetric matrices, which are obviously determined by $N(N+1)/2$ scalars. For each matrix $B \in \mathbb{M}^N$, we have a unique representation

$$B = B_{sym} + B_{skew}, \tag{2.1}$$

where $B_{sym} := \frac{1}{2}\left(B + B^t\right) \in \mathbb{S}^N_{sym}$ and $B_{skew} := \frac{1}{2}\left(B - B^t\right) \in \mathbb{S}^N_{skew}$. In the sequel, we will always identify each matrix $B \in \mathbb{M}^N$ with its decomposition in the form (2.1).

Let $p, q \in [1, \infty]$ be given real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $L^{2p}\left(\Omega; \mathbb{S}^N_{skew}\right)$ be the normed space of measurable $2p$-integrable functions whose values are skew-symmetric matrices.

Let $A(x)$ and $B(x)$ be given matrices such that $A, B \in L^{2p}(\Omega; \mathbb{S}^N_{skew})$. We say that these matrices are related by the binary relation $\preceq$ on the set $L^{2p}(\Omega; \mathbb{S}^N_{skew})$ (in symbols, $A(x) \preceq B(x)$ a.e. in Ω), if

$$\mathcal{L}^N\left(\cup_{i=1}^N \cup_{j=i+1}^N \left\{x \in \Omega \ : \ a_{ij}(x) > b_{ij}(x)\right\}\right) = 0. \tag{2.2}$$

Here, $\mathcal{L}^N(E)$ denotes the N-dimensional Lebesgue measure of $E \subset \mathbb{R}^N$ defined on the completed borelian σ-algebra.

Let $\alpha \in \mathbb{R}$ be a fixed positive value. Let ζ_{ad} and β be given $L^1(\Omega)$-functions satisfying the properties

$$\beta > \zeta_{ad} \geq 0 \ \text{ a.e. in } \ \Omega, \quad \zeta_{ad}^{-1} \in L^{2q}(\Omega) \quad \text{for } \ q = p/(p-1), \ p \geq 1, \tag{2.3}$$

$$\beta, \zeta_{ad} : \Omega \to \mathbb{R}^1_+ \ \text{ are smooth functions along the boundary } \ \partial\Omega, \tag{2.4}$$

$$\zeta_{ad} = \beta = \alpha \quad \text{on} \quad \partial\Omega. \tag{2.5}$$

By $\mathfrak{M}^{\beta}_{\zeta_{ad}}(\Omega)$ we denote the set of all matrices $A = [a_{i\,j}(\cdot)] \in L^1(\Omega; \mathbb{S}^N_{sym})$ such that

$$\zeta_{ad} I \leq A(x) \leq \beta(x) I \quad \text{a. e. in } \ \Omega \tag{2.6}$$

Here, I is the identity matrix in $\mathbb{R}^{N\times N}$, and (2.6) should be considered in the sense of quadratic forms defined by $(A\xi, \xi)_{\mathbb{R}^N}$ for $\xi \in \mathbb{R}^N$. Therefore, condition (2.6) implies the following inequalities:

$$\text{if } \ A \in \mathfrak{M}^{\beta}_{\zeta_{ad}}(\Omega), \ \text{then } \ \|A\|_{L^1(\Omega; \mathbb{S}^N_{sym})} \leq \|\beta\|_{L^1(\Omega)} < +\infty, \tag{2.7}$$

$$\zeta_{ad}(x)\|\xi\|^2_{\mathbb{R}^N} \leq (A(x)\xi, \xi)_{\mathbb{R}^N} \quad \text{a. e. in } \ \Omega, \ \forall \xi \in \mathbb{R}^N \tag{2.8}$$

$$\|A^{-1/2}(x)\xi\|^2_{\mathbb{R}^N} \leq \zeta_{ad}^{-1}(x)\|\xi\|^2_{\mathbb{R}^N} \quad \text{a. e. in } \ \Omega, \ \forall \xi \in \mathbb{R}^N, \tag{2.9}$$

and, therefore,

$$A^{-1/2} \in L^{4q}(\Omega; \mathbb{S}^N_{sym}) \quad \text{and} \quad \|A^{-1/2}\|_{L^{4q}(\Omega; \mathbb{S}^N_{sym})} \leq \sqrt{\|\zeta_{ad}^{-1}\|_{L^{2q}(\Omega)}}. \tag{2.10}$$

To each matrix $A \in \mathfrak{M}^{\beta}_{\zeta_{ad}}(\Omega) \subset L^1(\Omega; \mathbb{S}^N_{sym})$ we will associate two weighted Sobolev spaces: $W_A(\Omega) = W(\Omega; A\,dx)$ and $H_A(\Omega) = H(\Omega; A\,dx)$, where $W_A(\Omega)$ is the set of functions $y \in W^{1,1}_0(\Omega)$ for which the norm

$$\|y\|_A = \Big(\int_\Omega \big(y^2 + (\nabla y, A(x)\nabla y)_{\mathbb{R}^N}\,\big)\,dx\Big)^{1/2} \tag{2.11}$$

is finite, and $H_A(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $W_A(\Omega)$-norm. It is well-known that due to the inequality (2.8) the space $W_A(\Omega)$ is complete with respect to the norm $\|\cdot\|_A$ (see [11]). It is clear that $H_A(\Omega) \subset W_A(\Omega)$, and $W_A(\Omega)$, $H_A(\Omega)$ are Hilbert spaces.

For our further analysis, we make use of the following observation.

Remark 2.1. If $N \ge 2$ and there exists a value $\nu \in \big(\frac{N}{2}, +\infty\big)$ such that $u^{-\nu} \in L^1(\Omega)$, then the expressions (for more details see [4, pp.46]):

$$||y||_{1,H_u} = \left[\int_\Omega u\,|\nabla y|^2\,dx\right]^{1/2} \quad \text{and} \tag{2.12}$$

$$\|y\|_{2,H_u} = \Big(\int_\Omega \big(y^2 + u\,|\nabla y|^2\big)\,dx\Big)^{1/2} \tag{2.13}$$

can be considered as equivalent norms on $H_u := \mathrm{cl}_{\|\cdot\|_{2,H_u}} C^\infty_0(\Omega)$. Moreover, in this case the embedding $H_u \hookrightarrow L^2(\Omega)$ is compact. Taking this fact and definition of the class $\mathfrak{M}^{\beta}_{\zeta_{ad}}(\Omega)$ into account, we deduce that the norm $\|\cdot\|_A$, given by (2.11), is equivalent to the following one

$$\|y\|_{1,A} = \Big(\int_\Omega (\nabla y, A(x)\nabla y)_{\mathbb{R}^N}\ dx\Big)^{1/2} \tag{2.14}$$

on $H_A(\Omega)$ provided $A \in \mathfrak{M}^{\beta}_{\zeta_{ad}}(\Omega)$, $\zeta^{-1}_{ad} \in L^{2q}(\Omega)$ with $q = p/(p-1)$, where

$$p \in [1,\infty) \ \text{ if } \ N \in \{2,3\}, \quad \text{and} \quad 1 \le p \le \frac{N}{N-4} \ \text{ if } \ N \ge 4. \tag{2.15}$$

Indeed, since the conditions (2.15) implies the fulfilment of inequality $q = p/(p-1) > N/4$, it follows that $\nu := 2q \in \big(\frac{N}{2}, +\infty\big)$ and $\zeta^{-\nu}_{ad} \in L^1(\Omega)$.

Let $A = A_{sym} + A_{skew} \in L^1\big(\Omega; \mathbb{M}^N\big)$ be a given matrix matrix such that $A_{skew} \in L^{2p}\big(\Omega; \mathbb{S}^N_{skew}\big)$. In what follows, we associate with A the bilinear skew-symmetric form

$$\Phi(y,v)_A = \int_\Omega \big(\nabla v, A_{skew}(x)\nabla y\big)_{\mathbb{R}^N}\,dx, \quad \forall\, y, v \in W_{A_{sym}}(\Omega),$$

and introduce the matrix $C(x) \in \mathbb{S}^N_{skew}$ following the rule

$$C(x) = A^{-1/2}_{sym}(x) A_{skew}(x) A^{-1/2}_{sym}(x) \quad \text{a.e. in } \ \Omega. \tag{2.16}$$

It is easy to see that $C \in L^2(\Omega;\mathbb{S}^N_{skew})$. Indeed, by the Cauchy-Bunyakowsky inequality and estimate (2.10), we have

$$\begin{aligned}\|C\|^2_{L^2(\Omega;\mathbb{S}^N_{skew})} &\le \int_\Omega \|A_{skew}\|^2_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)}\|A_{sym}^{-\frac{1}{2}}\|^4_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)}\,dx\\ &\le \left(\int_\Omega \|A_{skew}\|^{2p}_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)}\,dx\right)^{1/p}\left(\int_\Omega \|A_{sym}^{-\frac{1}{2}}\|^{4q}_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)}\,dx\right)^{1/q}\\ &= \|A_{skew}\|^2_{L^{2p}(\Omega;\mathbb{S}^N_{skew})}\,\|A_{sym}^{-\frac{1}{2}}\|^4_{L^{4q}(\Omega;\mathbb{S}^N_{sym})}\\ &\le \|\zeta_{ad}^{-1}\|^2_{L^{2q}(\Omega)}\|A_{skew}\|^2_{L^{2p}(\Omega;\mathbb{S}^N_{skew})} < +\infty. \end{aligned} \tag{2.17}$$

Hence, the form $\Phi(y,v)_A$ is unbounded on $W_{A_{sym}}(\Omega)$, in general.

However, if we temporary assume that $C \in L^\infty(\Omega;\mathbb{S}^N_{skew})$, then the bilinear form $\Phi(\cdot,\cdot)_A$ is obviously bounded on $W_{A_{sym}}(\Omega)$. In this case we have

$$\begin{aligned}\left|\int_\Omega \left(\nabla\varphi, A_{skew}\nabla y\right)_{\mathbb{R}^N}dx\right|^2 &= \left|\int_\Omega \left(A_{sym}^{\frac{1}{2}}\nabla\varphi, \left[A_{sym}^{-\frac{1}{2}}A_{skew}A_{sym}^{-\frac{1}{2}}\right]A_{sym}^{\frac{1}{2}}\nabla y\right)_{\mathbb{R}^N}dx\right|^2\\ &\le \|C\|_{L^\infty(\Omega;\mathbb{S}^N_{skew})}\int_\Omega |A_{sym}^{\frac{1}{2}}\nabla\varphi|^2_{\mathbb{R}^N}\,dx\int_\Omega |A_{sym}^{\frac{1}{2}}\nabla y|^2_{\mathbb{R}^N}\,dx\\ &\le \|C\|_{L^\infty(\Omega;\mathbb{S}^N_{skew})}\,\|\varphi\|_{A_{sym}}\|y\|_{A_{sym}}.\end{aligned}$$

In order to deal with the case $C \notin L^\infty(\Omega;\mathbb{S}^N_{skew})$, we notice that the value $\Phi(y,v)_A$ is always finite provided $y \in W_{A_{sym}}(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. Indeed,

$$\begin{aligned}|\Phi(y,v)_A|^2 := \left|\int_\Omega \left(\nabla\varphi, A_{skew}\nabla y\right)_{\mathbb{R}^N}dx\right|^2 &\le \|\varphi\|^2_{C^1(\overline{\Omega})}\left(\int_\Omega |A_{skew}\nabla y|_{\mathbb{R}^N}\,dx\right)^2\\ &\le \|\varphi\|^2_{C^1(\overline{\Omega})}\int_\Omega \left\|A_{skew}A_{sym}^{-\frac{1}{2}}\right\|^2_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)}dx\int_\Omega \left|A_{sym}^{\frac{1}{2}}\nabla y\right|^2_{\mathbb{R}^N}dx\\ &\le \|\varphi\|^2_{C^1(\overline{\Omega})}\int_\Omega \left\|A_{skew}\right\|^2_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)}\zeta_{ad}^{-1}\,dx\int_\Omega \left(\nabla y, A_{sym}\nabla y\right)_{\mathbb{R}^N}dx\\ &\le \|\varphi\|^2_{C^1(\overline{\Omega})}\|y\|^2_A\left(\int_\Omega \|A_{skew}\|^{2p}_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)}\,dx\right)^{1/p}\left(\int_\Omega \zeta_{ad}^{-q}\,dx\right)^{1/q}\\ &\le \|\varphi\|^2_{C^1(\overline{\Omega})}\|y\|^2_A|\Omega|^{1/2q}\,\|\zeta_{ad}^{-1}\|_{L^{2q}(\Omega)}\,\|A_{skew}\|^2_{L^{2p}(\Omega;\mathbb{S}^N_{skew})} < +\infty.\end{aligned}$$

Hence, if $C \in L^2(\Omega;\mathbb{S}^N_{skew})$ then the integral $\int_\Omega \left(\nabla\varphi, A_{skew}(x)\nabla y\right)_{\mathbb{R}^N}dx$ is well defined for every $y \in W_{A_{sym}}(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. Taking this fact into account, we set

$$[y,\varphi]_A = \int_\Omega \left(\nabla\varphi, A_{skew}(x)\nabla y\right)_{\mathbb{R}^N}dx = \int_\Omega \left(A_{sym}^{1/2}\nabla\varphi, C(x)A_{sym}^{1/2}\nabla y\right)_{\mathbb{R}^N}dx,$$
$$\forall\, y \in W_{A_{sym}}(\Omega),\ \forall\,\varphi \in C_0^\infty(\Omega),$$

where the matrix C is defined by (2.16), and introduce of the following notion.

Let $V_{A_{sym}}(\Omega)$ be some intermediate space with $H_{A_{sym}}(\Omega) \subseteq V_{A_{sym}}(\Omega) \subseteq W_{A_{sym}}(\Omega)$.

Definition 2.1. Let $A = A_{sym} + A_{skew} \in L^1\big(\Omega; \mathbb{M}^N\big)$ be a given matrix such that $A_{skew} \in L^{2p}\big(\Omega; \mathbb{S}^N_{skew}\big)$. We say that an element $y \in V_{A_{sym}}(\Omega)$ belongs to the set $D(V_{A_{sym}})$ if

$$\left|\int_\Omega \big(\nabla\varphi, A_{skew}\nabla y\big)_{\mathbb{R}^N}\, dx\right| \le c(y,A)\left(\int_\Omega \varphi^2\, dx + \int_\Omega (\nabla\varphi, A_{sym}\nabla\varphi)_{\mathbb{R}^N}\, dx\right)^{1/2}$$
$$= c(y,A)\|\varphi\|_{A_{sym}}, \quad \forall\, \varphi \in C_0^\infty(\Omega) \tag{2.18}$$

with some constant c depending on y and A.

As a result, if $y \in D(V_{A_{sym}})$ then the mapping $\varphi \mapsto [y,\varphi]_A$ can be defined for all $\varphi \in H_{A_{sym}}(\Omega)$ using (2.18) and the standard rule

$$[y,\varphi]_A = \lim_{\varepsilon\to 0}\, [y,\varphi_\varepsilon]_A, \tag{2.19}$$

where $\{\varphi_\varepsilon\}_{\varepsilon>0} \subset C_0^\infty(\Omega)$ and $\varphi_\varepsilon \to \varphi$ strongly in $H_{A_{sym}}(\Omega)$ (it is the case where we essentially use the fact that $C_0^\infty(\Omega)$ is dense in $H_{A_{sym}}(\Omega)$). In particular, if $y \in D(H_{A_{sym}})$, then we can define the value $[y,y]_A$ and this one is finite for every $y \in D(H_{A_{sym}})$, although the "integrand" $\big(\nabla y, A_{skew}\nabla y\big)_{\mathbb{R}^N}$ needs not be integrable on Ω, in general.

Let $f : \Omega \to \mathbb{R}$ be a function of $L^1(\Omega)$. We define

$$TV(f) := \int_\Omega |Df| = \sup\Big\{ \int_\Omega f\,(\nabla,\varphi)_{\mathbb{R}^N}\, dx \ : \ \varphi = (\varphi_1,\dots,\varphi_N) \in C_0^1(\Omega;\mathbb{R}^N),\ |\varphi(x)| \le 1 \text{ for } x\in\Omega \Big\},$$

where $(\nabla,\varphi)_{\mathbb{R}^N} = \sum_{i=1}^N \frac{\partial \varphi_i}{\partial x_i}$.

According to the Radon-Nikodym theorem, if $TV(f) < +\infty$ then the distribution Df is a measure and there exist a vector-valued function $\nabla f \in [L^1(\Omega)]^N$ and a measure $D_s f$, singular with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N\lfloor\Omega$ restricted to Ω, such that $Df = \nabla f \mathcal{L}^N\lfloor\Omega + D_s f$.

Definition 2.2. A function $f \in L^1(\Omega)$ is said to have a bounded variation in Ω if $TV(f) < +\infty$. By $BV(\Omega)$ we denote the space of all functions in $L^1(\Omega)$ with bounded variation, i.e. $BV(\Omega) = \big\{ f \in L^1(\Omega) \ : \ TV(f) < +\infty \big\}$.

Under the norm $\|f\|_{BV(\Omega)} = \|f\|_{L^1(\Omega)} + TV(f)$, $BV(\Omega)$ is a Banach space. For our further analysis, we need the following properties of BV-functions (see [5]):

Proposition 2.1. Let $\{f_k\}_{k=1}^\infty$ be a sequence in $BV(\Omega)$ strongly converging to some f in $L^1(\Omega)$ and satisfying condition $\sup_{k\in\mathbb{N}} TV(f_k) < +\infty$. Then

$$f \in BV(\Omega) \quad \text{and} \quad TV(f) \le \liminf_{k\to\infty} TV(f_k)$$

and for every bounded sequence $\{f_k\}_{k=1}^\infty \subset BV(\Omega)$ there exists a subsequence, still denoted by f_k, and a function $f \in BV(\Omega)$ such that $f_k \to f$ in $L^1(\Omega)$.

Let $I_k : \mathbb{U}_k \times \mathbb{Y}_k \to \overline{\mathbb{R}}$ be a cost functional, $\mathbb{Y}_k$ be a space of states, and $\mathbb{U}_k$ be a space of controls. Let $\min\{I_k(u,y) \,:\, (u,y) \in \Xi_k\}$ be a parameterized OCP, where

$$\Xi_k \subset \{(u_k, y_k) \in \mathbb{U}_k \times \mathbb{Y}_k \,:\, u_k \in U_k,\ I_k(u_k, y_k) < +\infty\}$$

is a set of all admissible pairs linked by some state equation. Hereinafter we always associate to such OCP the corresponding constrained minimization problem:

$$(\mathrm{CMP_k}) \ : \qquad \left\langle \inf_{(u,y)\in\,\Xi_k} I_k(u,y) \right\rangle. \tag{2.20}$$

Since the sequence of constrained minimization problems (2.20) lives in variable spaces $\mathbb{U}_k \times \mathbb{Y}_k$, we assume that there exists a Banach space $\mathbb{U} \times \mathbb{Y}$ with respect to which a convergence in the scale of spaces $\{\mathbb{U}_k \times \mathbb{Y}_k\}_{k\in\mathbb{N}}$ is well defined (for the details, we refer to [12, 20]). In the sequel, we use the following notation for this convergence $(u_k, y_k) \stackrel{\mu}{\longrightarrow} (u,y)$ in $\mathbb{U}_k \times \mathbb{Y}_k$. Moreover, we assume that every bounded sequence in variable space $\mathbb{U}_k \times \mathbb{Y}_k$ is sequentially compact with respect to the μ-convergence.

In order to study the asymptotic behavior of a family of $(\mathrm{CMP_k})$, the passage to the limit in (2.20) as the parameter k tends to $+\infty$ has to be realized. The expression “passing to the limit”means that we have to find a kind of “limit cost functional” I and “limit set of constraints” Ξ with a clearly defined structure such that the limit object $\left\langle \inf_{(u,y)\in\Xi} I(u,y)\right\rangle$ may be interpreted as some OCP.

Following the scheme of the direct variational convergence [12], we adopt the following definition for the convergence of minimization problems in variable spaces.

Definition 2.3. A problem $\left\langle \inf_{(u,y)\in\Xi} I(u,y)\right\rangle$ is the variational μ-limit of the sequence (2.20) as $k \to \infty$, if and only if the following conditions are satisfied:

(d) If sequences $\{k_n\}_{n\in\mathbb{N}}$ and $\{(u_n, y_n)\}_{n\in\mathbb{N}}$ are such that $k_n \to 0$ as $n \to \infty$, $(u_n, y_n) \in \Xi_{k_n}$ $\forall\, n \in \mathbb{N}$, and $(u_n, y_n) \stackrel{\mu}{\longrightarrow} (u,y)$ in $\mathbb{U}_{k_n} \times \mathbb{Y}_{k_n}$, then

$$(u,y) \in \Xi; \quad I(u,y) \le \liminf_{n\to\infty} I_{k_n}(u_n, y_n); \tag{2.21}$$

(dd) For every $(u,y) \in \Xi \subset \mathbb{U} \times \mathbb{Y}$, there are an integer $k^0 > 0$ and a sequence $\{(u_k, y_k)\}_{k\in\mathbb{N}}$ (called a Γ-realizing sequence) such that

$$(u_k, y_k) \in \Xi_k,\ \forall\, k \ge k^0, \quad (u_k, y_k) \stackrel{\mu}{\longrightarrow} (\widehat{u}, \widehat{y}) \ \text{ in } \ \mathbb{U}_k \times \mathbb{Y}_k, \tag{2.22}$$

$$I(u,y) \ge \limsup_{k\to\infty} I_k(u_k, y_k). \tag{2.23}$$

Then the following result takes place [12].

Theorem 2.1. *Assume that the constrained minimization problem*

$$\left\langle \inf_{(u,y)\in\Xi_0} I_0(u,y)\right\rangle \tag{2.24}$$

is the variational μ-limit of sequence (2.20) in the sense of Definition 2.3 and this problem has a nonempty set of solutions

$$\Xi_0^{opt} := \left\{ (u^0, y^0) \in \Xi_0 \ : \ I_0(u^0, y^0) = \inf_{(u,y)\in\Xi_0} I_0(u,y) \right\}.$$

For every $k \in \mathbb{N}$, let $(u_k^0, y_k^0) \in \Xi_k$ be a minimizer of I_k on the corresponding set Ξ_k. If the sequence $\{(u_k^0, y_k^0)\}_{k\in\mathbb{N}}$ is relatively compact with respect to the μ-convergence in variable spaces $\mathbb{U}_k \times \mathbb{Y}_k$, then there exists a pair $(u^0, y^0) \in \Xi_0^{opt}$ such that

$$(u_k^0, y_k^0) \xrightarrow{\mu} (u^0, y^0) \quad in \ \ \mathbb{U}_k \times \mathbb{Y}_k, \tag{2.25}$$

$$\inf_{(u,y)\in\Xi_0} I_0(u,y) = I_0\left(u^0, y^0\right) = \lim_{k\to\infty} I_k(u_k^0, y_k^0) = \lim_{k\to\infty} \inf_{(u_k,y_k)\in\Xi_k} I_k(u_k, y_k). \tag{2.26}$$

3. Weak Convergence in Variable L^2-Spaces Associated with $\mathbb{S}^N_{sym}$-Valued Radon Measures

By a nonnegative Radon measure on Ω we mean a nonnegative Borel measure which is finite on every compact subset of Ω. The space of all nonnegative Radon measures on Ω will be denoted by $M_+(\Omega)$. According to the Riesz theory, each Radon measure $\mu \in M_+(\Omega)$ can be interpreted as an element of the dual of the space $C_0(\Omega)$ of all continuous functions with compact support. Let $M(\Omega; \mathbb{S}^N_{sym})$ denote the space of all $\mathbb{S}^N_{sym}$-valued Borel measures. Then $\mu = [\mu_{ij}] \in M(\Omega; \mathbb{S}^N_{sym})$ $\Leftrightarrow \mu_{ij} \in C_0'(\Omega)$, $i, j = 1, \dots, N$.

Let μ and the sequence $\{\mu_k\}_{k\in\mathbb{N}}$ be matrix-valued Radon measures. We say that $\{\mu_k\}_{k\in\mathbb{N}}$ weakly-$*$ converges to μ in $M(\Omega; \mathbb{S}^N_{sym})$ if

$$\lim_{k\to\infty} \int_\Omega \varphi \cdot d\mu_k = \int_\Omega \varphi \cdot d\mu \quad \forall\, \varphi \in C_0(\Omega; \mathbb{S}^N_{sym}).$$

A typical example of such measures is

$$d\mu_k = A_k(x)\, dx, \quad d\mu = A(x)\, dx, \tag{3.1}$$

$$\text{where} \ \ A_k, A \in L^1(\Omega; \mathbb{S}^N_{sym}) \ \ \text{and} \ \ A_k \to A \ \ \text{in} \ \ L^1(\Omega; \mathbb{S}^N_{sym}). \tag{3.2}$$

Hereinafter we suppose that the measures μ and $\{\mu_k\}_{k\in\mathbb{N}}$ are defined by (3.1)–(3.2). Then $\mu_k \xrightarrow{*} \mu$ in $M(\Omega; \mathbb{S}^N_{sym})$. Further, we will use $L^2(\Omega, A\, dx)^N$ to denote the Hilbert space of measurable vector-valued functions $f \in \mathbb{R}^N$ on Ω such that

$$\|f\|_{L^2(\Omega, A\, dx)^N} = \left(\int_\Omega (f, A(x) f)_{\mathbb{R}^N}\, dx \right)^{1/2} < +\infty.$$

We say that a sequence $\left\{ v_k \in L^2(\Omega, A_k\, dx)^N \right\}_{k\in\mathbb{N}}$ in is bounded if

$$\limsup_{k\to\infty} \int_\Omega (v_k, A_k(x) v_k)_{\mathbb{R}^N}\, dx < +\infty.$$

Definition 3.1. A bounded sequence $\left\{v_k \in L^2(\Omega, A_k\, dx)^N\right\}_{k\in\mathbb{N}}$ is weakly convergent to a function $v \in L^2(\Omega, A\, dx)^N$ in variable space $L^2(\Omega, A_k\, dx)^N$ if

$$\lim_{k\to\infty} \int_\Omega (\varphi, A_k(x) v_k)_{\mathbb{R}^N}\, dx = \int_\Omega (\varphi, A(x) v)_{\mathbb{R}^N}\, dx \quad \forall\, \varphi \in C_0^\infty(\Omega)^N. \tag{3.3}$$

Definition 3.2. A sequence $\left\{v_k \in L^2(\Omega, A_k\, dx)^N\right\}_{k\in\mathbb{N}}$ is said to be strongly convergent to a function $v \in L^2(\Omega, A\, dx)^N$ if

$$\lim_{k\to\infty} \int_\Omega (b_k, A_k(x) v_k)_{\mathbb{R}^N}\, dx = \int_\Omega (b, A(x) v)_{\mathbb{R}^N}\, dx \tag{3.4}$$

whenever $b_k \rightharpoonup b$ in $L^2(\Omega, A_k\, dx)^N$ as $k \to \infty$.

Remark 3.1. Note that in the case $A_k \equiv A$, Definitions 3.1–3.2 leads to the usual notion of convergence in weighted Hilbert space $L^2(\Omega, A\, dx)^N$.

The main properties of the weak and strong convergences in $L^p(\Omega, d\mu_\varepsilon)$ can be expressed as follows (see [11] for the details):

Proposition 3.1. If a sequence $\left\{v_k \in L^2(\Omega, A_k\, dx)^N\right\}_{k\in\mathbb{N}}$ is bounded and the condition (3.2) holds true, then it contains a weakly convergent subsequence in $L^2(\Omega, A_k\, dx)^N$.

Proposition 3.2. If the sequence $\left\{v_k \in L^2(\Omega, A_k\, dx)^N\right\}_{k\in\mathbb{N}}$ converges weakly to $v \in L^2(\Omega, A\, dx)^N$ and the condition (3.2) holds true, then

$$\liminf_{k\to\infty} \int_\Omega (v_k, A_k(x) v_k)_{\mathbb{R}^N}\, dx \geq \int_\Omega (v, A(x) v)_{\mathbb{R}^N}\, dx. \tag{3.5}$$

Proposition 3.3. Assume the condition (3.2) holds true. Then the weak convergence of a sequence $\left\{v_k \in L^2(\Omega, A_k\, dx)^N\right\}_{k\in\mathbb{N}}$ to $v \in L^2(\Omega, A\, dx)^N$ and

$$\lim_{k\to\infty} \int_\Omega (v_k, A_k(x) v_k)_{\mathbb{R}^N}\, dx = \int_\Omega (v, A(x) v)_{\mathbb{R}^N}\, dx \tag{3.6}$$

are equivalent to the strong convergence of $\{v_k\}_{k\in\mathbb{N}}$ in $L^2(\Omega, A_k\, dx)^N$ to $v \in L^2(\Omega, A\, dx)^N$.

In what follows, we make use of the following result.

Lemma 3.1. *Let $A^{skew} \in L^{2p}(\Omega; \mathbb{S}^N_{skew})$ be a given matrix, and let $\left\{A_k^{sym}\right\}_{k\in\mathbb{N}} \subset \mathfrak{M}^\beta_{\zeta_{ad}}(\Omega)$ be a sequence such that*

$$A_k^{sym} \to A_0^{sym} \quad in \quad L^1(\Omega; \mathbb{S}^N_{sym}). \tag{3.7}$$

Let $\varphi \in C_0^\infty(\Omega)$ be an arbitrary test function. Then

$$\mathbf{v}_k := \left(A_k^{sym}\right)^{-1} \nabla\varphi \to \left(A_0^{sym}\right)^{-1} \nabla\varphi =: \mathbf{v}_0 \quad and$$
$$\mathbf{w}_k := \left(A_k^{sym}\right)^{-1} A^{skew} \nabla\varphi \to \left(A_0^{sym}\right)^{-1} A^{skew} \nabla\varphi =: \mathbf{w}_0$$

strongly in variable space $L^2(\Omega, A_k^{sym}\, dx)^N$ as $k \to \infty$.

Proof. Let $\varphi \in C_0^\infty(\Omega)$ be an arbitrary test function. Since

$$\|\mathbf{v}_k\|^2_{L^2(\Omega,A_k^{sym}\,dx)^N} = \int_\Omega \left(\mathbf{v}_k, A_k^{sym}\mathbf{v}_k\right)_{\mathbb{R}^N}\,dx = \int_\Omega \left(\nabla\varphi, \left(A_k^{sym}\right)^{-1}\nabla\varphi\right)_{\mathbb{R}^N}\,dx$$
$$\le \int_\Omega |\nabla\varphi|^2_{\mathbb{R}^N}\zeta_{ad}^{-1}\,dx \le \|\varphi\|^2_{C^1(\Omega)}|\Omega|^{\frac{2p}{p+1}}\|\zeta_{ad}^{-1}\|_{L^{2q}(\Omega)} < +\infty,$$

$$\|\mathbf{w}_k\|^2_{L^2(\Omega,A_k^{sym}\,dx)^N} = \int_\Omega \left(\mathbf{w}_k, A_k^{sym}\mathbf{w}_k\right)_{\mathbb{R}^N}\,dx$$
$$= \int_\Omega \left(A^{skew}\nabla\varphi, \left(A_k^{sym}\right)^{-1}A^{skew}\nabla\varphi\right)_{\mathbb{R}^N}\,dx$$
$$\le \int_\Omega \zeta_{ad}^{-1}|A^{skew}\nabla\varphi|^2\,dx \le \|\varphi\|^2_{C^1(\Omega)}\|\zeta_{ad}^{-1}\|_{L^q(\Omega)}\|A^{skew}\|^2_{L^{2p}(\Omega;\mathbb{S}^N_{skew})} < +\infty$$

and, for each $\psi \in C_0^\infty(\Omega)$,

$$\begin{aligned}\int_\Omega \left(\nabla\psi, A_k^{sym}\mathbf{v}_k\right)_{\mathbb{R}^N}\,dx &= \int_\Omega \left(\nabla\psi, A_k^{sym}\left(A_k^{sym}\right)^{-1}\nabla\varphi\right)_{\mathbb{R}^N}\,dx\\ &= \int_\Omega (\nabla\psi,\nabla\varphi)_{\mathbb{R}^N}\,dx = \int_\Omega (\nabla\psi, A_0^{sym}\mathbf{v}_0)_{\mathbb{R}^N}\,dx,\\ \int_\Omega \left(\nabla\psi, A_k^{sym}\mathbf{w}_k\right)_{\mathbb{R}^N}\,dx &= \int_\Omega \left(\nabla\psi, A_k^{sym}\left(A_k^{sym}\right)^{-1}A^{skew}\nabla\varphi\right)_{\mathbb{R}^N}\,dx\\ &= \int_\Omega \left(\nabla\psi, A^{skew}\nabla\varphi\right)_{\mathbb{R}^N}\,dx = \int_\Omega (\nabla\psi, A_0^{sym}\mathbf{w}_0)_{\mathbb{R}^N}\,dx,\end{aligned} \tag{3.8}$$

it follows that the sequences $\{\mathbf{v}_k\}_{k\in\mathbb{N}}$ and $\{\mathbf{w}_k\}_{k\in\mathbb{N}}$ are bounded and weakly convergent in variable space $L^2(\Omega, A_k^{sym}\,dx)^N$ to vector-valued functions $\mathbf{v}_0 \in L^2(\Omega, A_0^{sym}\,dx)^N$ and $\mathbf{w}_0 \in L^2(\Omega, A_0^{sym}\,dx)^N$, respectively.

In order to show that the sequence $\{\mathbf{v}_k\}_{k\in\mathbb{N}}$ is strongly convergent to $\mathbf{v}_0 := (A_0^{sym})^{-1}\nabla\varphi$, we make use of Proposition 3.3. Following this assertion, it is enough to prove the equality

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega \left(\mathbf{v}_k, A_k^{sym}\mathbf{v}_k\right)_{\mathbb{R}^N}\,dx &= \lim_{k\to\infty}\int_\Omega \left(\nabla\varphi, \left(A_k^{sym}\right)^{-1}\nabla\varphi\right)_{\mathbb{R}^N}\,dx\\ &= \int_\Omega \left(\nabla\varphi, (A_0^{sym})^{-1}\nabla\varphi\right)_{\mathbb{R}^N}\,dx = \int_\Omega (\mathbf{v}_0, A_0^{sym}\mathbf{v}_0)_{\mathbb{R}^N}\,dx.\end{aligned} \tag{3.9}$$

In view of estimate $\left(\nabla\varphi, \left(A_k^{sym}\right)^{-1}\nabla\varphi\right)_{\mathbb{R}^N} \le \|\varphi\|^2_{C^1(\Omega)}\zeta_{ad}^{-1} < +\infty, \quad \forall\, k\in\mathbb{N}$, the sequence $\left\{\left(\nabla\varphi, \left(A_k^{sym}\right)^{-1}\nabla\varphi\right)_{\mathbb{R}^N}\right\}_{k\in\mathbb{N}}$ ie equi-integrable. On the other hand, property (3.7) implies that, within a subsequence, we have the pointwise convergence $\left(A_k^{sym}\right)^{-1} \to (A_0^{sym})^{-1}$ almost everywhere in Ω. Hence, up to a subsequence,

$$\left(\nabla\varphi, \left(A_k^{sym}\right)^{-1}\nabla\varphi\right)_{\mathbb{R}^N} \to \left(\nabla\varphi, (A_0^{sym})^{-1}\nabla\varphi\right)_{\mathbb{R}^N} \quad \text{a.e. in } \Omega.$$

Thus, the equality (3.9) is a direct consequence of Lebesgue Dominated Theorem, and hence,

$$\left(A_k^{sym}\right)^{-1}\nabla\varphi \to (A_0^{sym})^{-1}\nabla\varphi \ \text{ strongly in } \ L^2(\Omega, A_k^{sym}\,dx)^N \quad \forall\,\varphi\in C_0^\infty(\Omega).$$

Following the same arguments, it can be shown that

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega \left(\mathbf{w}_k, A_k^{sym}\mathbf{w}_k\right)_{\mathbb{R}^N}\,dx &= -\lim_{k\to\infty}\int_\Omega \Big(\nabla\varphi, A^{skew}\left(A_k^{sym}\right)^{-1}A^{skew}\nabla\varphi\Big)_{\mathbb{R}^N}\,dx\\ &= -\int_\Omega \Big(\nabla\varphi, A^{skew}\,(A_0^{sym})^{-1}A^{skew}\nabla\varphi\Big)_{\mathbb{R}^N}\,dx = \int_\Omega (\mathbf{w}_0, A_0^{sym}\mathbf{w}_0)_{\mathbb{R}^N}\,dx.\end{aligned}$$

Combining this fact with relation (3.8), by Proposition 3.3 we have:

$$\left(A_k^{sym}\right)^{-1}A^{skew}\nabla\varphi \to (A_0^{sym})^{-1}A^{skew}\nabla\varphi \ \text{ strongly in } \ L^2(\Omega, A_k^{sym}\,dx)^N$$

$\forall\,\varphi\in C_0^\infty(\Omega)$. The proof is complete. □

4. Setting of the Optimal Control Problem

Let $p\ge 1$ be a given exponent and let $f:\Omega\to\mathbb{R}^N$ be a vector-valued function such that $f\in L^{4p/(p+1)}(\Omega;\mathbb{R}^N)$. Let $M\in\mathbb{S}^N_{sym}$ be a constant matrix satisfying the condition

$$(M\xi,\xi)_{\mathbb{R}^N}\ge m\|\xi\|^2_{\mathbb{R}^N} \ \text{ for some } \ m>0.$$

The optimal control problem we consider in this paper is to minimize the discrepancy (tracking error) between a given distribution $y_d\in L^2(\Omega)$ and a solution y of the Dirichlet boundary value problem for the linear elliptic equation

$$-\operatorname{div}\big(A(x)\nabla y\big)+a_0(x)y = -\operatorname{div} f \quad \text{in} \quad \Omega, \tag{4.1}$$

$$y=0 \quad \text{on} \quad \partial\Omega. \tag{4.2}$$

by choosing an appropriate matrix-valued control $A(x)=A_{sym}(x)+A_{skew}(x)$. Here, $a_0\in L^\infty(\Omega)$ is a given function such that $a_0(x)\ge\alpha>0$ almost everywhere in Ω.

More precisely, we are concerned with the following OCP

$$\text{Minimize } I(A,y)=\|y-y_d\|^2_{L^2(\Omega)}+\int_\Omega(\nabla y, A_{sym}\nabla y)_{\mathbb{R}^N}\,dx \tag{4.3}$$

$$\text{subject to the constraints (4.1)–(4.2) with } A\in\mathfrak{A}_{ad}\subset L^1(\Omega;\mathbb{M}^N). \tag{4.4}$$

In order to define the class of admissible controls $\mathfrak{A}_{ad}$, we begin with some preliminaries. Let $A^*, A^{**}\in L^{2p}(\Omega;\mathbb{S}^N_{skew})$ be given nonzero matrices such that $A^*\preceq A^{**}$ a.e. in Ω, let c be a given positive constant, and let Q be a nonempty convex compact subset of $L^{2p}(\Omega;\mathbb{S}^N_{skew})$ such that the null matrix $A\equiv[0]$ belongs to Q. Further we make use of the following sets

$$U_{a,1}=\left\{A=[a_{i\,j}]\in L^1(\Omega;\mathbb{S}^N_{sym})\;\big|\,TV(a_{ij})\le c,\;1\le i\le j\le N\right\},\tag{4.5}$$

$$U_{b,1}=\left\{A=[a_{i\,j}]\in L^1(\Omega;\mathbb{S}^N_{sym})\;|\;A\in\mathfrak{M}^{\beta}_{\zeta_{ad}}(\Omega),\;\int_\Omega A(x)\,dx=M\right\},\tag{4.6}$$

$$U_{a,2}=\left\{A=[a_{i\,j}]\in L^{2p}(\Omega;\mathbb{S}^N_{skew})\;\big|\;A^*(x)\preceq A(x)\preceq A^{**}(x)\text{ a.e. in }\;\Omega\right\},\tag{4.7}$$

$$U_{b,2}=\left\{A=[a_{i\,j}]\in L^{2p}(\Omega;\mathbb{S}^N_{skew})\;\big|\;A\in Q\right\}.\tag{4.8}$$

Remark 4.1. Hereinafter we assume that

$$\mathfrak{A}_{ad,1}:=U_{a,1}\cap U_{b,1}\neq\emptyset\quad\text{ and }\quad\mathfrak{A}_{ad,2}:=U_{a,2}\cap U_{b,2}\neq\emptyset,$$

and, hence, the set $\mathfrak{A}_{ad}:=\mathfrak{A}_{ad,1}\oplus\mathfrak{A}_{ad,2}$ is nonempty. Moreover, it is easy to see that for a given $A^*,A^{**}\in L^{2p}(\Omega;\mathbb{S}^N_{skew})$, we can always guarantee the fulfilment of condition $\mathfrak{A}_{ad}\neq\emptyset$ by an appropriate choice of functions $\zeta_{ad}\in L^1(\Omega)$ and $\beta\in L^1(\Omega)$, a matrix $M\in\mathbb{S}^N_{sym}$, and a compact subset Q.

Definition 4.1. We say that a matrix $A=A_{sym}+A_{skew}$ is an admissible control to the Dirichlet boundary value problem (4.1)–(4.2) (it is written as $A\in\mathfrak{A}_{ad}$) if $A_{sym}\in\mathfrak{A}_{ad,1}$ and $A_{skew}\in\mathfrak{A}_{ad,2}$.

For our further analysis, we use of the following results.

Proposition 4.1. The set $\mathfrak{A}_{ad}$ is convex and sequentially compact with respect to the strong topology of $L^1(\Omega;\mathbb{M}^N)$.

Proof. Let $\left\{A_k=A_k^{sym}+A_k^{skew}\right\}_{k\in\mathbb{N}}\subset\mathfrak{A}_{ad}$ be an arbitrary sequence of admissible controls. Since

$$\mathfrak{A}_{ad}=\mathfrak{A}_{ad,1}\oplus\mathfrak{A}_{ad,2},\quad\mathfrak{A}_{ad,1}\subset BV(\Omega;\mathbb{S}^N_{sym}),$$
$$\mathfrak{A}_{ad,2}\subset U_{b,2},\;\text{ and }\;U_{b,2}\text{ is a compact in }\;L^{2p}(\Omega;\mathbb{S}^N_{skew}),$$

it follows by the compactness of BV-functions (see Proposition 2.1) that there exist matrices $A_0^{sym}\in BV(\Omega;\mathbb{S}^N_{sym})$ and $A_0^{skew}\in U_{b,2}\subset L^{2p}(\Omega;\mathbb{S}^N_{skew})$ such that within a subsequence

$$A_k^{sym}\to A_0^{sym}\quad\text{in }\;L^1(\Omega;\mathbb{S}^N_{sym}),\tag{4.9}$$

$$A_k^{skew}\to A_0^{skew}\quad\text{in }\;L^{2p}(\Omega;\mathbb{S}^N_{skew}),\tag{4.10}$$

$$\text{and }\;A_k\to A_0:=A_0^{sym}+A_0^{skew}\;\text{ almost everywhere in }\;\Omega.\tag{4.11}$$

Combining these facts with definition of the binary relation $\preceq$ (see (2.4)), we arrive at the conclusion: $A_0^{skew}\in U_{a,2}$, $A_0^{skew}\in U_{b,2}$, and $A_0^{sym}\in U_{a,1}$. Hence, it remains to show the condition $A_0^{sym}\in U_{b,1}$. With that in mind we make use of the following observation.

By the initial suppositions, we have $A_k^{sym} \in \mathfrak{M}_{\zeta_{ad}}^{\beta}(\Omega)$ for all $k \in \mathbb{N}$. Hence, in view of L^1-convergence $A_k^{sym} \to A_0^{sym}$, we may assume that, up to a subsequence, $A_k^{sym} \to A_0^{sym}$ almost everywhere in Ω. Since $A_k(x) \geq \zeta_{ad} I$ a. e. in Ω, it follows that

$$\begin{aligned} M &= \lim_{k\to\infty} \int_\Omega A_k^{sym}(x)\,dx = \int_\Omega A_0^{sym}(x)\,dx, \\ A_0^{sym}(x) &= \lim_{k\to\infty} A_k^{sym}(x) \leq \beta(x) I \quad \text{a. e. in } \ \Omega, \\ \zeta_{ad}(x) I &\leq \lim_{k\to\infty} A_k^{sym}(x) = A_0^{sym}(x) \quad \text{a. e. in } \ \Omega. \end{aligned}$$

Thus, $A_0^{sym} \in U_{b,1}$. As a result, we have

$$A_k := A_k^{sym} + A_k^{skew} \to A_0^{sym} + A_0^{skew} =: A_0 \ \text{ in } \ L^1(\Omega;\mathbb{M}^N).$$

and $A_0 \in \mathfrak{A}_{ad}$. Since the convexity of $\mathfrak{A}_{ad}$ is obviously valid, this concludes the proof. □

Definition 4.2. We say that a function $y = y(A, f)$ is a weak solution to the boundary value problem (4.1)–(4.2) for a fixed admissible control $A = A_{sym} + A_{skew} \in \mathfrak{A}_{ad}$ and given distribution $f \in L^{4p/(p+1)}(\Omega;\mathbb{R}^N)$ if $y \in W_{A_{sym}}(\Omega)$ and the integral identity

$$\int_\Omega \left[(\nabla\varphi, A_{sym}\nabla y)_{\mathbb{R}^N} + a_0 y\varphi \right] dx + \int_\Omega \left(\nabla\varphi, A_{skew}\nabla y\right)_{\mathbb{R}^N} dx = \int_\Omega (f,\nabla\varphi)_{\mathbb{R}^N}\ dx \tag{4.12}$$

holds for each $\varphi \in C_0^\infty(\Omega)$.

We note that by the initial assumptions and Hölder's inequality, this definition makes a sense because $(A_{skew}\nabla y) \in L^1(\Omega;\mathbb{R}^N)$ for each $y \in W_{A_{sym}}(\Omega)$. Indeed,

$$\begin{aligned} &\int_\Omega |A_{skew}\nabla y|_{\mathbb{R}^N}\,dx \leq \int_\Omega \|A_{skew} A_{sym}^{-1/2}\|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)}\,|A_{sym}^{1/2}\nabla y|_{\mathbb{R}^N}\,dx \\ &\leq \left(\int_\Omega \|A_{skew}\|^2_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)} \|A_{sym}^{-1/2}\|^2_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)}\,dx\right)^{1/2} \left(\int_\Omega (\nabla y, A_{sym}\nabla y)_{\mathbb{R}^N}\,dx\right)^{1/2} \\ &\leq \left(\int_\Omega \|A_{skew}\|^{2p}_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)}\,dx\right)^{1/2p} \left(\int_\Omega \|A_{sym}^{-1/2}\|^{2q}_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)}\,dx\right)^{1/2q} \|y\|_{A_{sym}} \\ &\leq |\Omega|^{1/4q}\,\|A_{skew}\|_{L^{2p}(\Omega;\mathbb{S}^N_{skew})}\,\|A_{sym}^{-1/2}\|_{L^{4q}(\Omega;\mathbb{S}^N_{skew})}\,\|y\|_{A_{sym}}. \end{aligned}$$

On the other hand, Definition 4.2 gives another motivation to introduce the set $D(W_{A_{sym}})$.

Proposition 4.2. Let $A = A_{sym} + A_{skew} \in \mathfrak{A}_{ad}$ and $f \in L^{4p/(p+1)}(\Omega;\mathbb{R}^N)$ be given distributions. Let $y \in V_{A_{sym}}(\Omega)$ be a weak solution to the boundary value problem (4.1)–(4.2) for some intermediate space $V_{A_{sym}}(\Omega)$ with $H_{A_{sym}}(\Omega) \subseteq V_{A_{sym}}(\Omega) \subseteq W_{A_{sym}}(\Omega)$. Then $y \in D(V_{A_{sym}})$.

Proof. In order to prove this assertion it is enough to rewrite the integral identity (4.12) in the form

$$[y,\varphi]_A = -\int_\Omega \big(A_{sym}\nabla y, \nabla\varphi\big)_{\mathbb{R}^N}\,dx - \int_\Omega a_0 y\varphi\,dx + \int_\Omega (f,\nabla\varphi)_{\mathbb{R}^N}\,dx \tag{4.13}$$

and apply Hölder's inequality to the right-hand side of (3.4). As a result, we have

$$\Big|\int_\Omega \big(A_{sym}\nabla y, \nabla\varphi\big)_{\mathbb{R}^N}\,dx\Big| \le \left(\int_\Omega |A_{sym}^{1/2}\nabla y|^2_{\mathbb{R}^N}\,dx\right)^{1/2}\left(\int_\Omega |A_{sym}^{1/2}\nabla \varphi|^2_{\mathbb{R}^N}\,dx\right)^{1/2}$$
$$\le \|y\|_{A_{sym}}\|\varphi\|_{A_{sym}},$$

$$\Big|\int_\Omega a_0 y\varphi\,dx\Big| \le \|a_0\|_{L^\infty(\Omega)}\|y\|_{L^2(\Omega)}\left(\|\varphi\|^2_{L^2(\Omega)} + \int_\Omega |A_{sym}^{1/2}\nabla \varphi|^2_{\mathbb{R}^N}\,dx\right)^{\frac{1}{2}}$$
$$\le \|a_0\|_{L^\infty(\Omega)}\|y\|_{A_{sym}}\|\varphi\|_{A_{sym}},$$

$$\begin{aligned}\left|\int_\Omega (f,\nabla\varphi)_{\mathbb{R}^N}\,dx\right| &\le \left(\int_\Omega |A_{sym}^{-1/2} f|^2_{\mathbb{R}^N}\,dx\right)^{1/2}\left(\int_\Omega |A_{sym}^{1/2}\nabla \varphi|^2_{\mathbb{R}^N}\,dx\right)^{1/2}\\
&= \left(\int_\Omega |A_{sym}^{-1/2} f|^2_{\mathbb{R}^N}\,dx\right)^{1/2}\left(\int_\Omega (\nabla\varphi, A_{sym}\nabla \varphi)_{\mathbb{R}^N}\,dx\right)^{1/2}\\
&\le \left(\int_\Omega \|A_{sym}^{-1/2}\|^{4q}_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)}\,dx\right)^{1/4q}\left(\int_\Omega |f|^{4p/p+1}_{\mathbb{R}^N}\,dx\right)^{(p+1)/4p}\\
&\times\left(\int_\Omega (\nabla\varphi, A_{sym}\nabla \varphi)_{\mathbb{R}^N}\,dx\right)^{1/2}\\
&\le \sqrt{\|\zeta_{ad}^{-1}\|_{L^{2q}(\Omega)}}\ \|f\|_{L^{4p/p+1}(\Omega;\mathbb{R}^N)}\|\varphi\|_{A_{sym}},\end{aligned} \tag{4.14}$$

and, hence,

$$|[y,\varphi]_A| \le \left(\big(1+\|a_0\|_{L^\infty(\Omega)}\big)\,\|y\|_{A_{sym}} + \sqrt{\|\zeta_{ad}^{-1}\|_{L^{2q}(\Omega)}}\ \|f\|_{L^{4p/p+1}(\Omega;\mathbb{R}^N)}\right)$$
$$\times\left(\int_\Omega (\nabla\varphi, A_{sym}\nabla \varphi)_{\mathbb{R}^N}\,dx\right)^{1/2} \le c(y,A)\|\varphi\|_{A_{sym}},\quad \forall\,\varphi\in C_0^\infty(\Omega).$$

□

As estimate (4.14) obviously indicates, Proposition 4.2 can be specified as follows.

Corollary 4.1. *Let $A = A_{sym} + A_{skew} \in \mathfrak{A}_{ad}$ be an arbitrary admissible control, and let f be a distribution such that $A_{sym}^{-1/2} f \in L^2(\Omega;\mathbb{R}^N)$. Let $y \in V_{A_{sym}}(\Omega)$ be a weak solution to the boundary value problem (4.1)–(4.2). Then $y \in D(V_{A_{sym}})$.*

Remark 4.2. Due to Proposition 4.2, Definition 4.2 can be reformulated as follows: y is a weak solution to the problem (4.1)–(4.2) for a given control $A = A_{sym} + A_{skew} \in \mathfrak{A}_{ad}$, if and only if $y \in D(W_{A_{sym}})$ and

$$\int_\Omega \left[\left(\nabla\varphi, A_{sym}\nabla y\right)_{\mathbb{R}^N} + a_0 y\varphi \right] dx + [y,\varphi]_A = \int_\Omega (f, \nabla\varphi)_{\mathbb{R}^N}\, dx \quad \forall\, \varphi \in C_0^\infty(\Omega). \tag{4.15}$$

Moreover, as follows from (2.19), (4.15), and (4.14), if a weak solution to the problem (3.1)–(3.2) belongs to the space $H_{A_{sym}}(\Omega)$ then it satisfies the energy equality

$$\int_\Omega \left(\nabla y, A_{sym}\nabla y\right)_{\mathbb{R}^N} dx + \int_\Omega a_0 y^2\, dx + [y,y]_A = \int_\Omega (f, \nabla y)_{\mathbb{R}^N}\, dx. \tag{4.16}$$

It is worth to notice that the original boundary value problem (4.1)–(4.2) is ill-possed and, in general. Moreover, in view of definition of the set $\mathfrak{M}^\beta_{\zeta_{ad}}(\Omega)$, the existence of a weak solution to (4.1)–(4.2) for fixed $A \in \mathfrak{A}_{ad}$ and $f \in L^{4p/(p+1)}(\Omega;\mathbb{R}^N)$ seems to be an open question. This means that there are no reasons to expect that for every admissible given data $f \in L^{4p/(p+1)}(\Omega;\mathbb{R}^N)$ and $A \in \mathfrak{A}_{ad}$, this problem admits at least one weak solution $y \in W_{A_{sym}}(\Omega)$ in the sense of Definition 4.2. At the same time, even if a weak solution to the above problem exists, the question about its uniqueness remains open. Indeed, because of the properties of function ζ_{ad}, we face with the problem of density of smooth functions $C_0^\infty(\Omega)$ in $W_{A_{sym}}(\Omega)$. As was indicated in [21], there exists a diagonal matrix-valued function $A(x) = \rho(x)I$ with $\rho \geq \zeta_{ad}$ such that the subspace $C_0^\infty(\Omega)$ is not dense in $W_{A_{sym}}(\Omega)$. Therefore, even if we assume that we have two weak solutions $y_1(A,f), y_2(A,f) \in W_{A_{sym}}(\Omega)$ such that $y_1(A,f) \neq y_2(A,f)$,

$$\int_\Omega \left(\nabla y_k, A_{skew}\nabla y_k\right)_{\mathbb{R}^N} dx = 0, \quad k = 1,2$$

(this is always true for $A_{skew} \in L^\infty\big(\Omega;\mathbb{S}^N_{skew}\big)$), and each of these solutions satisfies the corresponding energy equality

$$\int_\Omega \left(\nabla y_k, A_{sym}\nabla y_k\right)_{\mathbb{R}^N} dx + \int_\Omega y_k^2\, dx = \int_\Omega (f, \nabla y_k)_{\mathbb{R}^N}\, dx, \quad k = 1,2, \tag{4.17}$$

then the element $y = \left(y_1(A,f) + y_2(A,f)\right)/2$ is a weak solution to (4.1)–(3.2) too, but it does not satisfy (4.17) in general. Thus, the degenerate boundary value problem (4.1)–(4.2) can admit weak solutions which do not satisfy energy equality. For more details and other types of solutions to degenerate equations we refer to [20, 21].

On the other hand, as it follows from the definition of the bilinear form $[y,\varphi]_A$, the value $[y,y]_A$ may not of constant sign for all $y \in D(W_{A_{sym}})$. Hence, even if the relation $H_{A_{sym}}(\Omega) = W_{A_{sym}}(\Omega)$ is valid, the energy equality (4.16) does not allow us to derive a reasonable a priory estimate in $\|\cdot\|_A$-norm for the weak solutions.

Thus, the mapping $A \mapsto y(A, f)$ can be multivalued, in general (see [7] for the details).

Taking these observations into account, we restrict of our analysis to the following set of admissible solutions for the original optimal control problem. Namely, we indicate the set

$$\Xi = \left\{(A, y) \;\middle|\; A \in \mathfrak{A}_{ad},\ y \in W_{A_{sym}}(\Omega),\ (A, y) \text{ are related by (4.15)} \right\}. \quad (4.18)$$

The characteristic feature of this set is the fact that for different admissible controls $A \in \mathfrak{A}_{ad}$ the 'corresponding' weak solutions y belong to different weighted spaces. Moreover, we adopt the following hypothesis, which is mainly motivated by the previous reasonings.

Hypothesis A. *The set of admissible solutions Ξ is nonempty.*

We say that a pair $(A^0, y^0) \in L^1(\Omega; \mathbb{M}^N) \times W_{A^0_{sym}}(\Omega)$ is a weak optimal solution to the problem (4.3)–(4.4) on the set Ξ, if

$$(A^0, y^0) \in \Xi \text{ and } I(A^0, y^0) = \inf_{(A,y)\in\Xi} I(A, y). \quad (4.19)$$

Our next observation deals with some specification of the set of admissible controls $\mathfrak{A}_{ad}$. With that in mind we give a few auxiliary results.

Lemma 4.1. *Let $\left\{A_k = A_k^{sym} + A_k^{skew}\right\}_{k\in\mathbb{N}} \subset \mathfrak{A}_{ad}$ and $\{y_k \in W_{A_k^{sym}}\}_{k\in\mathbb{N}}$ be sequences such that*

$$A_k := A_k^{sym} + A_k^{skew} \to A_0^{sym} + A_0^{skew} =: A_0 \quad in \quad L^1(\Omega; \mathbb{M}^N), \quad (4.20)$$

$$y_k \rightharpoonup y \quad in \quad L^2(\Omega), \qquad \nabla y_k \rightharpoonup \mathbf{v} \quad in \quad L^2(\Omega, A_k^{sym}\, dx)^N. \quad (4.21)$$

Then $A_0 \in \mathfrak{A}_{ad}$, $y \in W_{A_0^{sym}}(\Omega)$, and $\nabla y = \mathbf{v}$.

Proof. In view of Proposition 4.1, it is enough to prove the equality $\nabla y = \mathbf{v}$. Taking into account the estimates

$$\begin{aligned}
&\int_\Omega |\nabla y_k|_{\mathbb{R}^N}\, dx = \int_\Omega |\left(A_k^{sym}\right)^{-1/2} \left(A_k^{sym}\right)^{1/2} \nabla y_k|_{\mathbb{R}^N}\, dx \\
&\quad \le \left(\int_\Omega \|\left(A_k^{sym}\right)^{-1/2}\|^2_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)}\, dx\right)^{1/2} \left(\int_\Omega \left(\nabla y_k, A_k^{sym}\nabla y_k\right)_{\mathbb{R}^N}\, dx\right)^{1/2} \\
&\quad \overset{\text{by } (4.21)_2 \text{ and } (2.9)}{\le} C \left(\int_\Omega \zeta_{ad}^{-1}\, dx\right)^{1/2} \le C|\Omega|^{\frac{p}{p+1}} \|\zeta_{ad}^{-1}\|^{\frac{1}{2}}_{L^{2q}(\Omega)} < +\infty, \\
&\int_\Omega |\nabla \mathbf{v}|_{\mathbb{R}^N}\, dx \le \|\mathbf{v}\|_{L^2(\Omega, A_0^{sym}\, dx)^N} |\Omega|^{\frac{p}{p+1}} \|\zeta_{ad}^{-1}\|^{\frac{1}{2}}_{L^{2q}(\Omega)} < +\infty,
\end{aligned}$$

we conclude that $\nabla y_k, \mathbf{v} \in L^1(\Omega)^N$ for all $k \in \mathbb{N}$.

Further, we make use of Lemma 3.1. Following this result, for each test function $\varphi \in C_0^\infty(\Omega)$, we have $\left(A_k^{sym}\right)^{-1} \nabla\varphi \to \left(A_0^{sym}\right)^{-1} \nabla\varphi$ strongly in variable space

$L^2(\Omega, A_k^{sym}\,dx)^N$. Then, the definition of the strong convergence in variable spaces implies $\forall\,\varphi \in C_0^\infty(\Omega)$

$$\lim_{k\to\infty}\int_\Omega (\nabla\varphi, \nabla y_k)_{\mathbb{R}^N}\,dx = \lim_{k\to\infty}\int_\Omega \left(\left(A_k^{sym}\right)^{-1}\nabla\varphi, A_k^{sym}\nabla y_k\right)_{\mathbb{R}^N}\,dx$$
$$\overset{\text{by }(4.21)_2,\ (3.4)}{=} \int_\Omega \left(\left(A_0^{sym}\right)^{-1}\nabla\varphi, A_0^{sym}\nabla y_k\right)_{\mathbb{R}^N}\,dx = \int_\Omega (\nabla\varphi, \mathbf{v})_{\mathbb{R}^N}\,dx.$$

Combining this fact with relation

$$\lim_{k\to\infty}\int_\Omega y_k\varphi\,dx = \int_\Omega y\varphi\,dx, \quad \forall\,\varphi\in C_0^\infty(\Omega),$$

we finally conclude: $y_k \rightharpoonup y$ in $L^1(\Omega)$, $\nabla y_k \rightharpoonup \mathbf{v}$ in $L^1(\Omega)^N$, and therefore, $\nabla y = \mathbf{v}$ and $y \in W_0^{1,1}(\Omega)$ by completeness of the Sobolev space $W_0^{1,1}(\Omega)$. To end the proof, it remains to observe that $y\in L^2(\Omega)$ and $\nabla y \in L^2(\Omega, A_0^{sym}\,dx)^N$. □

For our further analysis we temporary assume that the functions β and ζ_{ad} are extended to the whole space of $\mathbb{R}^N$, i.e.

$$\beta, \zeta_{ad} \in L^1_{loc}(\mathbb{R}^N),\ 0\le \zeta_{ad}(x) \le \beta(x) \text{ a.e. in } \Omega, \quad\text{and}\quad \zeta_{ad}^{-1}\in L^1_{loc}(\mathbb{R}^N),$$

and there exists a constant $C>0$ such that

$$\sup_{B\in\mathbb{R}^N}\left(\frac{1}{|B|}\int_B \beta\,dx\right)\left(\frac{1}{|B|}\int_B \zeta_{ad}^{-1}\,dx\right) \le C, \tag{4.22}$$

where B is a ball in $\mathbb{R}^N$.

Theorem 4.1 ([20]). *Assume the condition* (4.22) *holds true for some constant* $C>0$. *Then for each admissible control* $A = A_{sym} + A_{skew} \in \mathfrak{A}_{ad}$, *we have* $H_{A_{sym}}(\Omega) = W_{A_{sym}}(\Omega)$ *and, hence, every weak solution to the boundary value problem* (3.1)–(3.2) *satisfies the energy equality* (4.16).

We are now in a position to establish the main result of this section.

Theorem 4.2. *Assume that, for given threshold matrices* $A^*, A^{**}\in L^{2p}(\Omega;\mathbb{S}^N_{skew})$, *Hypothesis A is valid. Then the optimal control problem* (4.3)–(4.4) *admits at least one solution for all distributions* $f\in L^{4p/(p+1)}(\Omega;\mathbb{R}^N)$ *and* $y_d\in L^2(\Omega)$.

Proof. Since the original problem is regular and the cost functional for the given problem is bounded below on Ξ, it follows that there exists a minimizing sequence $\{(A_k,y_k)\}_{k\in\mathbb{N}} \subset \Xi$ such that $I(A_k,y_k) \xrightarrow[k\to\infty]{} I_{\min} \equiv \inf_{(A,y)\in\Xi} I(A,y) \ge 0$. Hence, $\sup_{k\in\mathbb{N}} I(A_k,y_k)\le C$, where the constant C is independent of k. Since

$$\sup_{k\in\mathbb{N}}\|y_k\|^2_{A_k^{sym}} = \int_\Omega \left(y_k^2 + \left(\nabla y_k, A_k^{sym}(x)\nabla y_k\right)_{\mathbb{R}^N}\right)dx$$
$$\le 2\sup_{k\in\mathbb{N}} I(A_k,y_k) + 2\|y_d\|^2_{L^2(\Omega)} \le 2\left(C + \|y_d\|^2_{L^2(\Omega)}\right),$$

in view of Propositions 3.1, 4.1, and Lemma 3.1, it follows that passing to a subsequence if necessary, we may assume the existence of a pair $(A_0, y_0) \in \mathfrak{A}_{ad} \times W_{A_0^{sym}}(\Omega)$ such that

$$
\begin{aligned}
A_k := A_k^{sym} + A_k^{skew} \to A_0^{sym} + A_0^{skew} =: A_0 \ \text{ in } \ L^1(\Omega; \mathbb{M}^N), \qquad (4.23)\\
A_k^{sym} \to A_0^{sym} \quad \text{in} \ \ L^1(\Omega; \mathbb{S}^N_{sym}), \qquad (4.24)\\
A_k^{skew} \to A_0^{skew} \quad \text{in} \ \ L^{2p}(\Omega; \mathbb{S}^N_{skew}), \qquad (4.25)\\
y_k \rightharpoonup y_0 \ \text{ in } \ L^2(\Omega), \qquad (4.26)\\
\nabla y_k \rightharpoonup \nabla y_0 \ \text{ in } \ L^2(\Omega, A_k^{sym}\, dx)^N, \qquad (4.27)\\
I(A_0, y_0) < +\infty. \qquad (4.28)
\end{aligned}
$$

Since $(A_k, y_k) \in \Xi$ for every $k \in \mathbb{N}$, it follows that the integral identity

$$
\int_\Omega \left[\left(\nabla\varphi, A_k^{sym} \nabla y_k\right)_{\mathbb{R}^N} + a_0 \varphi y_k \right] \, dx + \int_\Omega \left(\nabla\varphi, A_k^{skew} \nabla y_k\right)_{\mathbb{R}^N} dx = \int_\Omega (f, \nabla\varphi)_{\mathbb{R}^N} \, dx \quad (4.29)
$$

holds true for all $\varphi \in C_0^\infty(\Omega)$. In order to pass to the limit in (4.29), we note that

$$
\begin{aligned}
\int_\Omega \left(\nabla\varphi, A_k^{skew} \nabla y_k\right) dx = &-\int_\Omega \left(\left(A_k^{sym}\right)^{-\frac{1}{2}} \left(A_k^{skew} - A_0^{skew}\right) \nabla\varphi, \left(A_k^{sym}\right)^{\frac{1}{2}} \nabla y_k \right) dx \\
&- \int_\Omega \left(\left(A_k^{sym}\right)^{-\frac{1}{2}} A_0^{skew} \nabla\varphi, \left(A_k^{sym}\right)^{\frac{1}{2}} \nabla y_k \right) dx = I_{1,k} + I_{2,k}
\end{aligned}
$$

by the skew-symmetry property of A_k^{skew} and A_0^{skew}. Since

$$
\begin{aligned}
\lim_{k\to\infty} |I_{1,k}| \le & \lim_{k\to\infty} \int_\Omega \left| \left(A_k^{skew} - A_0^{skew}\right) \left(A_k^{sym}\right)^{-\frac{1}{2}} \nabla\varphi \right| \left| \left(A_k^{sym}\right)^{\frac{1}{2}} \nabla y_k \right| dx \\
\le & \lim_{k\to\infty} \left(\int_\Omega \|A_k^{skew} - A_0^{skew}\|^2 \| \left(A_k^{sym}\right)^{-\frac{1}{2}} \|^2 |\nabla\varphi|^2 \, dx \right)^{\frac{1}{2}} \\
& \times \lim_{k\to\infty} \left(\int_\Omega | \left(A_k^{sym}\right)^{\frac{1}{2}} \nabla y_k|^2 \, dx \right)^{\frac{1}{2}} \le \|\varphi\|_{C^1(\Omega)} \sup_{k\in\mathbb{N}} \|y_k\|_{A_k^{sym}} \\
& \times \lim_{k\to\infty} \left(\int_\Omega \|A_k^{skew} - A_0^{skew}\|^{2p} \, dx \right)^{\frac{1}{2p}} \left(\int_\Omega \zeta_{ad}^{-q} \, dx \right)^{\frac{1}{2q}} \\
\overset{\text{by (4)}}{\le} & \sqrt{2\left(C + \|y_d\|^2_{L^2(\Omega)}\right)} \|\varphi\|_{C^1(\Omega)} |\Omega|^{\frac{1}{4q}} \|\zeta_{ad}^{-1}\|^{\frac{1}{2}}_{L^{2q}(\Omega)} \\
& \times \lim_{k\to\infty} \|A_k^{skew} - A_0^{skew}\|_{L^{2p}(\Omega; \mathbb{S}^N_{skew})} \overset{\text{by (4.25)}}{=} 0, \qquad (4.30)
\end{aligned}
$$

and

$$\lim_{k\to\infty} I_{2,k} = -\lim_{k\to\infty}\int_\Omega \big(\big(A_k^{sym}\big)^{-1} A_0^{skew}\nabla\varphi, A_k^{sym}\nabla y_k\big)_{\mathbb{R}^N}\,dx$$
$$= -\int_\Omega \big((A_0^{sym})^{-1} A_0^{skew}\nabla\varphi, A_0^{sym}\nabla y_0\big)_{\mathbb{R}^N}\,dx = \int_\Omega \big(\nabla\varphi, A_0^{skew}\nabla y_0\big)_{\mathbb{R}^N}\,dx \tag{4.31}$$

by (4.27), Lemma 3.1, and definition of the strong convergence in variable spaces, it follows that

$$\int_\Omega \big(\nabla\varphi, A_k^{skew}\nabla y_k\big)_{\mathbb{R}^N}\,dx \longrightarrow \int_\Omega \big(\nabla\varphi, A_0^{skew}\nabla y_0\big)_{\mathbb{R}^N}\,dx \quad \text{as} \quad k\to\infty.$$

Taking this fact and property (4.28) into account, we can pass to the limit in (4.29). As a result, we obtain

$$\int_\Omega \big[\big(\nabla\varphi, A_0^{sym}\nabla y_0\big) + a_0\varphi y_0\big]\,dx + \int_\Omega \big(\nabla\varphi, A_0^{skew}\nabla y_0\big)\,dx = \int_\Omega (f,\nabla\varphi)\,dx.$$

that is, a function $y_0 = y(A_0, f)$ is a weak solution to the boundary value problem (4.1)–(4.2) for admissible control $A = A_0^{sym} + A_0^{skew} \in \mathfrak{A}_{ad}$. Hence, $y_0 \in D(W_{A_0^{sym}})$ by Proposition 4.2, and, therefore, (A_0, y_0) is an admissible pair to problem (4.3)–(4.4).

In remains to show that (A_0, y_0) is an optimal pair. Using conditions (4.26)–(4.28) and the property of lower semicontinuity of the norms $\|\cdot\|_{L^2(\Omega, A\,dx)^N}$ and $\|\cdot\|_{L^2(\Omega)}$ with respect to the the weak topologies of $L^2(\Omega, A\,dx)^N$ and $L^2(\Omega)$, respectively (see Proposition 3.2), we get

$$\liminf_{k\to\infty} \|y_k - y_d\|^2_{L^2(\Omega)} \ge \|y_0 - y_d\|^2_{L^2(\Omega)}\,,$$
$$\liminf_{k\to\infty} \int_\Omega \big(\nabla y_k, A_k^{sym}\nabla y_k\big)_{\mathbb{R}^N}\,dx \ge \int_\Omega (\nabla y_0, A_0^{sym}\nabla y_0)_{\mathbb{R}^N}\,dx.$$

Thus,

$$I(A_0, y_0) \ge \inf_{(A,y)\in\Xi} I(A,y) = \lim_{k\to\infty} I(A_k, y_k) \ge \liminf_{k\to\infty} I(A_k, y_k)$$
$$\ge \|y_0 - y_d\|^2_{L^2(\Omega)} + \int_\Omega (\nabla y_0, A_0^{sym}\nabla y_0)_{\mathbb{R}^N}\,dx = I(A_0, y_0),$$

and hence, the pair (A_0, y_0) is optimal for problem (4.3)–(4.4). The proof is complete. □

5. On variational solutions to OCP (4.3)–(4.4) and their approximation

The question we are going to discuss in this section is about some pathological properties that can be inherited by optimal pair to the problem (4.3)–(4.4) and other unexpected surprises concerning the approximation of the original OCP and its solutions.

To begin with, we show that the main assumption on the regularity property of OCP (4.3)–(4.4) in Theorem 4.2 (see Hypothesis A) can be eliminated due to the approximation approach. For instance, the conditions $\zeta_{ad} \in L^1(\Omega)$ and $\zeta_{ad}^{-1} \in L^{2q}(\Omega)$ ensure the existence of a sequence of scalar positive functions $\{\zeta_k\}_{k\in\mathbb{N}}$ such that $\zeta_k \in L^\infty(\Omega)$ for all $k \in \mathbb{N}$, and

$$\zeta_k \to \zeta_{ad} \text{ strongly in } L^1(\Omega), \quad L^\infty(\Omega) \ni \zeta_k^{-1} \to \zeta_{ad}^{-1} \text{ strongly in } L^{2q}(\Omega). \tag{5.1}$$

By analogy we can approximate the rest components $A^*, A^{**} \in L^{2p}(\Omega; \mathbb{S}^N_{skew})$ and $\beta \in L^1(\Omega)$. The simplest way to construct such sequences is to apply the procedure of direct smoothing (5.2)–(5.3), i.e. we can set $\zeta_k := (\zeta_{ad})_k$, where

$$(\zeta_{ad})_k = k^N \int_{\mathbb{R}^N} K\left(k(x-z)\right) \widehat{\zeta_{ad}}(z)\, dz, \tag{5.2}$$

and K is a positive compactly supported smooth function such that

$$K \in C_0^\infty(\mathbb{R}^N), \ \int_{\mathbb{R}^N} K(x)\, dx = 1, \ \text{ and } \ K(x) = K(-x). \tag{5.3}$$

Here, $\widehat{\ }$ is a non-zero extension operator such that

$$\widehat{\zeta_{ad}} = \zeta_{ad} \text{ in } \Omega, \quad \widehat{\zeta_{ad}} \in L^1_{loc}(\mathbb{R}^N), \quad \widehat{\zeta_{ad}}^{-1} \in L^{2q}_{loc}(\mathbb{R}^N). \tag{5.4}$$

As a result, the property $(5.1)_1$ is the direct consequence of the classical properties of smoothing. In order to prove the property $(5.1)_2$, we note that

$$(\zeta_{ad})_k(x) \le C \int_Q \widehat{\zeta_{ad}}(x + k^{-1}z)\, dz, \quad \text{a.e. in } \mathbb{R}^N, \ \forall\, k \in \mathbb{N},$$

where Q is the support of the smoothing kernel K and $K(x) \le C$ by (5.3). Hence, $(\zeta_{ad})_k \in L^1_{loc}(\mathbb{R}^N)$ for all $k \in \mathbb{N}$. Using the Cauchy inequality

$$\begin{aligned} 1 = \left(\int_{\mathbb{R}^N} K(z)\, dz\right)^2 &= \left(\int_{\mathbb{R}^N} \left[K(z)\widehat{\zeta_{ad}}(x + \frac{z}{k})\right]^{\frac{1}{2}} \left[K(z)\widehat{\zeta_{ad}}^{-1}(x + \frac{z}{k})\right]^{\frac{1}{2}} dz\right)^2 \\ &\le \left(\int_{\mathbb{R}^N} K(z)\widehat{\zeta_{ad}}(x + \frac{z}{k})\, dz\right)\left(\int_{\mathbb{R}^N} K(z)\widehat{\zeta_{ad}}^{-1}(x + \frac{z}{k})\, dz\right) = (\zeta_{ad})_k \left(\zeta_{ad}^{-1}\right)_k, \end{aligned}$$

we see that

$$(\zeta_{ad})_k^{-1} \le \left(\zeta_{ad}^{-1}\right)_k \tag{5.5}$$

and, therefore,

$$\begin{aligned}
(\zeta_{ad})_k^{-2q} &\le \left(\zeta_{ad}^{-1}\right)_k^{2q} = \left(\int_{\mathbb{R}^N} K(z)\widehat{\zeta_{ad}}^{-1}(x+\frac{z}{k})\,dz\right)^{2q}\\
&= \left(\int_{\mathbb{R}^N} K^{\frac{2q-1}{2q}}(z)K^{\frac{1}{2q}}(z)\widehat{\zeta_{ad}}^{-1}(x+\frac{z}{k})\,dz\right)^{2q}\\
&\le \left(\int_{\mathbb{R}^N} K(z)\,dz\right)^{2q-1}\left(\int_{\mathbb{R}^N} K(z)\widehat{\zeta_{ad}}^{-2q}(x+\frac{z}{k})\,dz\right)\\
&= \int_{\mathbb{R}^N} K(z)\widehat{\zeta_{ad}}^{-2q}(x+\frac{z}{k})\,dz = \left(\zeta_{ad}^{-2q}\right)_k \le C\int_Q \widehat{\zeta_{ad}}^{-2q}(x+\frac{z}{k})\,dz. \qquad (5.6)
\end{aligned}$$

Hence, $(5.4)_3$ implies that $(\zeta_{ad})_k^{-1} \in L^{2q}_{loc}(\mathbb{R}^N)$ for all $k \in \mathbb{N}$.

Since $(\zeta_{ad})_k \to \zeta_{ad}$ in $L^1(\Omega)$ as $k \to \infty$ by the classical properties of smoothing, we can suppose that $(\zeta_{ad})_k^{-2q}(x) \to \zeta_{ad}^{-2q}(x)$ almost everywhere in Ω. In the meantime the inequality (5.6) guarantees the equi-integrability of $\left\{(\zeta_{ad})_k^{-2q}\right\}_{k\in\mathbb{N}}$ because the sequence $\left\{\left(\zeta_{ad}^{-2q}\right)_k\right\}_{k\in\mathbb{N}}$, converging to ζ_{ad}^{-2q} strongly in $L^1(\Omega)$, possesses this property. As a result, Lebesgue's Theorem implies that $(\zeta_{ad})_k^{-2q} \to \zeta_{ad}^{-2q}$ in $L^1(\Omega)$ as $k \to \infty$, and so the proof of property $(5.1)_2$ is complete.

Before proceeding further, we give a few auxiliary results.

Lemma 5.1. *Let $f \in L^{2p}(\Omega)$ and $\{f_n\}_{n\in\mathbb{N}} \subset L^{2p}(\Omega)$ be such that $f_n \to f$ in $L^{2p}(\Omega)$ as $n \to \infty$. Then, for each positive integer $k \in \mathbb{N}$, we have*

$$(f_n)_k \to (f)_k \quad in \quad L^{2p}(\Omega) \quad as \quad n \to \infty, \qquad (5.7)$$

where

$$(f_n)_k := k^N \int_{\mathbb{R}^N} K\left(k(x-y)\right)\widetilde{f_n}(y)\,dy = k^N \int_\Omega K\left(k(x-y)\right) f_n(y)\,dy, \quad \forall\, n \in \mathbb{N}.$$

Proof. Taking into account the properties (5.3) of the kernel K, we get

$$\begin{aligned}
\|\,(f_n)_k - (f)_k\,\|^{2p}_{L^{2p}(\Omega)} &:= \int_\Omega \left(k^N\int_{\mathbb{R}^N} K\left(k(x-y)\right)\left(\widetilde{f_n}(y)-\widetilde{f}(y)\right)\,dy\right)^{2p}\,dx\\
&= \int_\Omega\left(\int_{\mathbb{R}^N} K^{\frac{2p-1}{2p}}(z)K^{\frac{1}{2p}}(z)\left(\widetilde{f_n}(x+\frac{z}{k})-\widetilde{f}(x+\frac{z}{k})\right)\,dz\right)^{2p}\,dx\\
&\le \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} K(z)\,dz\right)^{2p-1}\int_{\mathbb{R}^N} K(z)\left(\widetilde{f_n}(x+\frac{z}{k})-\widetilde{f}(x+\frac{z}{k})\right)^{2p}\,dz\,dx\\
&= \int_{\mathbb{R}^N} k^N\int_{\mathbb{R}^N} K\left(k(x-y)\right)\left(\widetilde{f_n}(y)-\widetilde{f}(y)\right)^{2p}\,dy\,dx\\
&= \int_{\mathbb{R}^N}\left(k^N\int_{\mathbb{R}^N} K\left(k(x-y)\right)\,dx\right)\left(\widetilde{f_n}(y)-\widetilde{f}(y)\right)^{2p}\,dy\\
&= \int_{\mathbb{R}^N}\left(\widetilde{f_n}(y)-\widetilde{f}(y)\right)^{2p}\,dy = \|f_n - f\|^{2p}_{L^{2p}(\Omega)} \to 0.
\end{aligned}$$

□

Lemma 5.2. *Let $f \in L^{2p}(\Omega)$ and $\{f_n\}_{n\in\mathbb{N}} \subset L^{2p}(\Omega)$ be such that $f_n \to f$ in $L^{2p}(\Omega)$ as $n \to \infty$. Let $\{k_n\}_{n\in\mathbb{N}}$ be a sequence of positive integers converging to $+\infty$ as $n \to \infty$. Then*

$$(f_n)_{k_n} \to f \quad \text{in} \quad L^{2p}(\Omega) \quad \text{as} \quad n \to \infty. \tag{5.8}$$

Proof. We define a doubly indexed family $\{a_{n,k}\}_{\substack{n\in\mathbb{N}\\k\in\mathbb{N}}}$ in $\mathbb{R}$ as follows

$$a_{n,k} = \|(f_n)_k - f\|^{2p}_{L^{2p}(\Omega)} := \int_\Omega \left| f(x) - k^N \int_{\mathbb{R}^N} K\left(k(x-y)\right) \widetilde{f_n}(y)\,dy \right|^{2p} dx.$$

Since

$$(g)_k \to g \quad \text{in} \quad L^{2p}(\Omega) \quad \text{as} \quad f \to \infty, \quad \forall\, g \in L^{2p}(\Omega), \tag{5.9}$$

by the classical properties of smoothing, and

$$\lim_{k\to\infty}\lim_{n\to\infty} a_{n,k} \overset{\text{by Lemma 5.1}}{=} \lim_{k\to\infty} \|(f)_k - f\|^{2p}_{L^{2p}(\Omega)} \overset{\text{by (5.9)}}{=} 0,$$
$$\lim_{n\to\infty}\lim_{k\to\infty} a_{n,k} \overset{\text{by (5.9)}}{=} \lim_{n\to\infty} \|f_n - f\|^{2p}_{L^{2p}(\Omega)} \overset{\text{by the initial assumptions}}{=} 0,$$

it follows that $\lim\limits_{k\to\infty}\left(\lim\limits_{n\to\infty} a_{n,k}\right) = \lim\limits_{n\to\infty} a_{n,k_n} = \lim\limits_{n\to\infty}\left(\lim\limits_{k\to\infty} a_{n,k}\right)$. □

Following the similar arguments, Lemma 5.2 can be specified to the following particular case.

Lemma 5.3. *Let $f \in L^1(\Omega)$ and $\{f_n\}_{n\in\mathbb{N}} \subset L^1(\Omega)$ be such that $f_n \to f$ in $L^1(\Omega)$ as $n \to \infty$. Let $\{k_n\}_{n\in\mathbb{N}}$ be a sequence of positive integers converging to $+\infty$ as $n \to \infty$. Then*

$$(f_n)_{k_n} \to f \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad n \to \infty. \tag{5.10}$$

Taking these results into account, we bring into consideration the following sequence of constrained minimization problems associated with the Steklov smoothing operator $(\cdot)_k$:

$$\left\{ \left\langle \inf_{(A,y)\in\Xi_k} I_k(A,y) \right\rangle, \quad k \to \infty \right\}. \tag{5.11}$$

Here,

$$I_k(A,y) := I(A,y) \quad \forall\,(A,y) \in L^1(\Omega;\mathbb{M}^N) \times W_{A^{sym}}(\Omega), \quad \forall\, k \in \mathbb{N}, \tag{5.12}$$

$$\Xi_k = \left\{ (A,y) \;\middle|\; \begin{array}{c} -\mathrm{div}\left(A^{sym}\nabla y + A^{skew}\nabla y\right) = -\mathrm{div}\, f \quad \text{in} \quad \Omega, \\ y = 0 \text{ on } \partial\Omega, \quad y \in W_{A^{sym}}(\Omega), \\ A = A^{sym} + A^{skew} \in \mathfrak{A}^k_{ad} = \mathfrak{A}^k_{ad,1} \oplus \mathfrak{A}^k_{ad,2}, \\ A^{sym} \in \mathfrak{A}^k_{ad,1} \ \text{ iff } \ \exists\, C^{sym} \in \mathfrak{A}_{ad,1} \ \text{ s.t. } \ A^{sym} = (C^{sym})_k\,, \\ A^{skew} \in \mathfrak{A}^k_{ad,2} \ \text{ iff } \ \exists\, C^{skew} \in \mathfrak{A}_{ad,2} \ \text{ s.t. } \ A^{skew} = \left(C^{skew}\right)_k. \end{array} \right\} \tag{5.13}$$

Before we will provide an accurate analysis of the optimal control problems (5.11), we describe in more details some topological properties of the sets $\mathfrak{A}^k_{ad,1}$ and $\mathfrak{A}^k_{ad,2}$. We begin with the following observation.

Remark 5.1. In view of definition of the sets $\mathfrak{A}^k_{ad,1}$ and $\mathfrak{A}^k_{ad,2}$, the condition $A \in \mathfrak{A}_{ad,1} = \mathfrak{A}^k_{ad,1} \oplus \mathfrak{A}^k_{ad,2}$ implies the existence of a certain matrix $C(x) = C^{sym}(x) + C^{skew}(x)$ (the so-called 'prototype' of A) such that $C^{sym} \in \mathfrak{A}_{ad,1}$, $C^{skew} \in \mathfrak{A}_{ad,2}$, and $A = (C^{sym})_k + \left(C^{skew}\right)_k$ whatever matrix A was chosen.

Lemma 5.4. *For every $k \in \mathbb{N}$ there exist positive constants α_k and γ_k such that $\gamma_k > \alpha_k$ and*

$$\alpha_k \|\xi\|^2_{\mathbb{R}^N} \le (A^{sym}(x)\xi, \xi)_{\mathbb{R}^N} \le \gamma_k \|\xi\|^2_{\mathbb{R}^N} \quad \textit{a. e. in} \;\; \Omega, \; \forall\, \xi \in \mathbb{R}^N, \tag{5.14}$$

$$TV\left(a^{sym}_{ij}\right) \le c, \quad 1 \le i \le j \le N, \tag{5.15}$$

$$\int_{\mathbb{R}^N} A^{sym}(x)\, dx = M \tag{5.16}$$

for each $A^{sym} \in \mathfrak{A}^k_{ad,1}$.

Proof. Let C^{sym} be an arbitrary element of the set $\mathfrak{A}_{ad,1}$. Since, $\mathfrak{A}_{ad,1} \subset \mathfrak{M}^{\beta}_{\zeta_{ad}}(\Omega)$, it follows that $\zeta_{ad}\|\xi\|^2_{\mathbb{R}^N} \le (C^{sym}(x)\xi, \xi)_{\mathbb{R}^N} \le \beta \|\xi\|^2_{\mathbb{R}^N}$ a. e. in Ω, $\forall\, \xi \in \mathbb{R}^N$. Hence, for any $k \in \mathbb{N}$, we have

$$(\zeta_{ad})_k \|\xi\|^2_{\mathbb{R}^N} \le ((C^{sym})_k \xi, \xi)_{\mathbb{R}^N} \le (\beta)_k \|\xi\|^2_{\mathbb{R}^N} \quad \text{a. e. in} \;\; \Omega, \; \forall\, \xi \in \mathbb{R}^N,$$

and, therefore, the constants α_k and γ_k in (5.14) can be defined as follows

$$\alpha_k = \inf_{x \in \Omega} (\zeta_{ad})_k(x), \qquad \gamma_k = \sup_{x \in \Omega} (\beta)_k(x).$$

In view of the initial assumptions (2.7)–(2.9) and definition of the Steklov smoothing operator $(\cdot)_k$, we have $(\beta)_k \in L^\infty(\Omega)$ and $(\zeta_{ad})^{-1}_k \in L^\infty(\Omega)$ (see (5.5)). Hence, α_k is a positive constant, and $\gamma_k < +\infty$.

As for the estimate (5.15), for an arbitrary $\varphi = (\varphi_1, \dots, \varphi_N) \in C^1_0(\Omega; \mathbb{R}^N)$ such that $|\varphi(x)| \le 1$ in Ω, and arbitrary matrix $A^{sym} = [a^{sym}_{ij}]^N_{i,j=1} \in \mathfrak{A}^k_{ad,1}$, we have

$$\begin{aligned}
TV(a^{sym}_{ij}) &= \sup_{|\varphi| \le 1} \left\{ \int_{\mathbb{R}^N} a^{sym}_{ij}(x) (\nabla, \varphi)_{\mathbb{R}^N}\, dx \right\} \\
&= \sup_{|\varphi| \le 1} \left\{ \int_{\mathbb{R}^N} \left(c^{sym}_{ij}\right)_k (x) (\nabla, \varphi)_{\mathbb{R}^N}\, dx \right\} \\
&= \sup_{|\varphi| \le 1} \left\{ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(z)\, \widetilde{c^{sym}_{ij}}(x + k^{-1}z)\, dz (\nabla, \varphi)_{\mathbb{R}^N}\, dx \right\} \\
&\le \int_{\mathbb{R}^N} K(z) \sup_{|\varphi| \le 1} \left\{ \int_{\mathbb{R}^N} \widetilde{c^{sym}_{ij}}(x + k^{-1}z) (\nabla, \varphi)_{\mathbb{R}^N}\, dx \right\} dz \\
&= \int_{\mathbb{R}^N} K(z)\, TV(\widetilde{c^{sym}_{ij}})(x + k^{-1}z)\, dz \le c \int_{\mathbb{R}^N} K(z)\; dz = c
\end{aligned}$$

Having applied the similar arguments, namely,

$$\begin{aligned}\int_{\mathbb{R}^N} A^{sym}(x)\,dx &= \int_{\mathbb{R}^N} \left(C^{sym}\right)_k(x)\,dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z)\,\widetilde{C}^{sym}(x+k^{-1}z)\,dz\,dx\\ &= k^N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K\left(k(x-y)\right)\widetilde{C}^{sym}(y)\,dy\,dx\\ &= \int_{\mathbb{R}^N}\left(k^N\int_{\mathbb{R}^N} K\left(k(x-y)\right)\right)dx\,\widetilde{C}^{sym}(y)\,dy = \int_\Omega \widetilde{C}^{sym}(y)\,dy = M,\end{aligned}$$

we arrive at the control constraint (5.16). □

Lemma 5.5. *$\mathfrak{A}^k_{ad,1}$ is a convex and sequentially compact set with respect to the strong topology of $L^1(\Omega;\mathbb{S}^N_{sym})$ for each $k\in\mathbb{N}$.*

Proof. Since the convexity of $\mathfrak{A}^k_{ad,1}$ immediately follows from the linearity of the smoothing operator $(\cdot)_k$, we concentrate on the compactness property of this set. Let $\{A^{sym}_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence in $\mathfrak{A}^k_{ad,1}$, and let $\{C^{sym}_n\}_{n\in\mathbb{N}}\subset\mathfrak{A}_{ad,1}$ be a sequence of its prototypes, that is, $A^{sym}_n(x)=\left(C^{sym}_n\right)_k(x)$ for all $n\in\mathbb{N}$. By Proposition 4.1, there exists a matrix $C^{sym}_0\in\mathfrak{A}_{ad,1}$ such that, within a subsequence, $C^{sym}_n\to C^{sym}_0$ in $L^1(\Omega;\mathbb{S}^N_{sym})$. As a result, Lemma 5.3 implies the strong convergence $A^{sym}_n\to A^{sym}_0$ in $L^1(\Omega;\mathbb{S}^N_{sym})$, where $A^{sym}_0=\left(C^{sym}_0\right)_k$ for a given $k\in\mathbb{N}$. □

We recall here that a sequence $\left\{\mathfrak{A}^k_{ad,1}\right\}_{k\in\mathbb{N}}$ of the subsets of $L^1(\Omega;\mathbb{S}^N_{sym})$ is said to be convergent to a closed set S in the sense of Kuratowski with respect to the strong topology of $L^1(\Omega;\mathbb{S}^N_{sym})$, if the following two properties hold:

(K_1) for every $A\in S$, there exists a sequence of matrices $\left\{A_k\in\mathfrak{A}^k_{ad,1}\right\}_{k\in\mathbb{N}}$ such that $A_k\to A$ in $L^1(\Omega;\mathbb{S}^N_{sym})$ as $k\to\infty$;

(K_2) if $\{k_n\}_{n\in\mathbb{N}}$ is a sequence of indices converging to $+\infty$, $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of symmetric matrices such that $A_n\in\mathfrak{A}^{k_n}_{ad,1}$ for each $n\in\mathbb{N}$, and $\{A_n\}_{n\in\mathbb{N}}$ strongly converges in $L^1(\Omega;\mathbb{S}^N_{sym})$ to some matrix A, then $A\in S$.

For the details we refer to [12]. As a result, we have the following result concerning asymptotic behaviour of the sequence $\left\{\mathfrak{A}^k_{ad,1}\right\}_{k\in\mathbb{N}}$.

Lemma 5.6. *The sequence of sets $\left\{\mathfrak{A}^k_{ad,1}\right\}_{k\in\mathbb{N}}$ converges to $\mathfrak{A}_{ad,1}$ as $k\to\infty$ in the sense of Kuratowski with respect to the strong topology of $L^1(\Omega;\mathbb{S}^N_{sym})$.*

Proof. In order to show that $S=\mathfrak{A}_{ad,1}$, we begin with the verification of (K_2)-item. Let $\{k_n\}_{n\in\mathbb{N}}$ be a given sequence of indices such that $k_n\to\infty$, and let $\left\{A_n\in\mathfrak{A}^{k_n}_{ad,1}\right\}_{n\in\mathbb{N}}$ be a sequence satisfying the property $A_n\to A$ in $L^1(\Omega;\mathbb{S}^N_{sym})$ as $n\to\infty$. By definition of the sets $\mathfrak{A}^k_{ad,1}$ and Proposition 4.1, there exists a sequence of prototypes $\{C_n^{sym}\}_{n\in\mathbb{N}} \subset \mathfrak{A}_{ad,1}$ and matrix $C_0^{sym} \in \mathfrak{A}_{ad,1}$ such that $A_n = (C_n^{sym})_{k_n}$ for all $n \in \mathbb{N}$ and, within a subsequence, $C_n^{sym} \to C_0^{sym}$ in $L^1(\Omega; \mathbb{S}^N_{sym})$. Then Lemma 5.3 guarantees the strong convergence $A_n \to C_0^{sym}$ in $L^1(\Omega; \mathbb{S}^N_{sym})$. As a result, we have $A = C_0^{sym}$ and, therefore, $A \in \mathfrak{A}_{ad,1}$. Since this assertion is valid for each L^1-converging subsequence of $\{C_n^{sym}\}_{n\in\mathbb{N}} \subset \mathfrak{A}_{ad,1}$, we finally get: the symmetric matrix A is L^1-limit for the entire sequence $\{C_n^{sym}\}_{n\in\mathbb{N}} \subset \mathfrak{A}_{ad,1}$.

It remains to verify the (K_1)-item. To this end, we fix an arbitrary symmetric matrix $A \in \mathfrak{A}_{ad,1}$ and construct the sequence $\left\{A_k \in \mathfrak{A}^k_{ad,1}\right\}_{k\in\mathbb{N}}$ as follows: $A_k = (A)_k$ for all $k \in \mathbb{N}$. Then $A_k \to A$ in $L^1(\Omega; \mathbb{S}^N_{sym})$ as $k \to \infty$ by main properties of the smoothing operator, and inclusions $A_k \in \mathfrak{A}^k_{ad,1}$, for each $k \in \mathbb{N}$, hold true by definition of the sets $\mathfrak{A}^k_{ad,1}$. □

Our next intention is to study topological and asymptotic properties of the sets $\mathfrak{A}^k_{ad,2}$.

Lemma 5.7. *For every $k \in \mathbb{N}$ each of the sets $\mathfrak{A}^k_{ad,2}$ is convex, sequentially compact with respect to the strong topology of $L^{2p}(\Omega; \mathbb{S}^N_{sym})$, and such that*

$$(A^*)_k(x) \preceq A(x) \preceq (A^{**})_k(x) \quad in \quad \Omega, \quad \forall A \in \mathfrak{A}^k_{ad,2}. \tag{5.17}$$

Proof. The convexity of $\mathfrak{A}^k_{ad,2}$ is a direct consequence of definition of the set $\mathfrak{A}_{ad,2}$ and the rule $(5.13)_6$. To prove the compactness property of this set let us consider an arbitrary sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathfrak{A}^k_{ad,2}$. Let $\left\{C_n^{skew}\right\}_{n\in\mathbb{N}} \subset \mathfrak{A}_{ad,2}$ be their prototypes, that is, $A_n(x) = \left(C_n^{skew}\right)_k(x)$ for all $n \in \mathbb{N}$. Since $\left\{C_n^{skew}\right\}_{n\in\mathbb{N}} \subset Q$, where Q is a nonempty convex compact subset of $L^{2p}(\Omega; \mathbb{S}^N_{skew})$, it follows that there exists a skew-symmetric matrix $C_0^{skew} \in Q$ such that, up to a subsequence, $C_n^{skew} \to C_0^{skew}$ in $L^{2p}(\Omega; \mathbb{S}^N_{skew})$. Then Lemma 5.1 implies the strong convergence $A_n \to A_0 := \left(C_0^{skew}\right)_k$ in $L^{2p}(\Omega; \mathbb{S}^N_{skew})$ for every $k \in \mathbb{N}$. It remains to note that in view of the definition of binary relation $\preceq$ (see (2.2)), for every $A = [a_{ij}] \in \mathfrak{A}_{ad,2}$, $i \in \{1, \dots, N\}$, $j \in \{i+1, \dots, N\}$, and $x \in \mathbb{R}^N$, we have

$$\begin{aligned}(a_{ij})_k(x) &:= k^N \int_{\mathbb{R}^N} K\left(k(x-y)\right) \widetilde{a_{ij}}(y)\,dy \\ &\le k^N \int_{\mathbb{R}^N} K\left(k(x-y)\right) \widetilde{a^{**}_{ij}}(y)\,dy = (a^{**}_{ij})_k(x), \quad \forall\, k \in \mathbb{N}.\end{aligned}$$

By analogy it can be shown that $(a^*_{ij})_k(x) \le (a_{ij})_k(x)$ in Ω. Hence, the restriction (5.17) holds true for each $k \in \mathbb{N}$ and $A \in \mathfrak{A}^k_{ad,2}$. The proof is complete. □

Lemma 5.8. *The sequence of sets $\left\{\mathfrak{A}^k_{ad,2}\right\}_{k\in\mathbb{N}}$ converges to $\mathfrak{A}_{ad,2}$ as $k \to \infty$ in the sense of Kuratowski with respect to the strong topology of $L^{2p}(\Omega; \mathbb{S}^N_{skew})$.*

Proof. We begin with the verification of (K_2)-property of the set $\mathfrak{A}_{ad,2}$ in the framework of definition of Kuratowski limit set with respect to the strong topology of $L^{2p}(\Omega;\mathbb{S}^N_{skew})$. Let $\{k_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of indices such that $k_n\to\infty$, and let $\left\{B_n\in\mathfrak{A}^{k_n}_{ad,2}\right\}_{n\in\mathbb{N}}$ be a sequence satisfying the property $B_n\to B$ in $L^{2p}(\Omega;\mathbb{S}^N_{skew})$ and, hence, up to a subsequence, $B_n(x)\to B(x)$ almost everywhere in Ω as $n\to\infty$. By Lemma 5.7, we have

$$(A^*)_{k_n}(x)\preceq B_n(x)\preceq (A^{**})_{k_n}(x)\quad \text{a.e. in }\ \Omega, \tag{5.18}$$

where $(A^*)_k\to A^*$ and $(A^{**})_k\to A^{**}$ strongly in $L^{2p}(\Omega;\mathbb{S}^N_{skew})$. Taking into account the fact that the binary relation $\preceq$ is a partial order, we can pass to the limit in relation (5.18) as $n\to\infty$ (in the sense of almost everywhere) and get $A^*(x)\preceq B(x)\preceq A^{**}(x)$ almost everywhere in Ω. In the meantime, closely following the arguments of the proof of Lemma 5.6 (see also Lemma 5.2), it can be shown that the matrix B is L^{2p}-limit of the corresponding sequence of prototypes $\left\{C^{skew}_n\right\}_{n\in\mathbb{N}}\subset Q\subset\mathfrak{A}_{ad,2}$, where $B_n=\left(C^{skew}_n\right)_{k_n}$ for all $n\in\mathbb{N}$. Since, Q is a compact set, it follows that $B\in Q$, and, therefore, $B\in\mathfrak{A}_{ad,2}$.

To verify the (K_1)-property, we fix an arbitrary skew-symmetric matrix $B\in\mathfrak{A}_{ad,2}$ and construct the sequence $\left\{B_k\in\mathfrak{A}^k_{ad,2}\right\}_{k\in\mathbb{N}}$ as follows: $B_k=(B)_k$ for all $k\in\mathbb{N}$. Then $B_k\to B$ in $L^{2p}(\Omega;\mathbb{S}^N_{skew})$ as $k\to\infty$ by main properties of the smoothing operator, and inclusions $B_k\in\mathfrak{A}^k_{ad,2}$, for each $k\in\mathbb{N}$, hold true by definition of the sets $\mathfrak{A}^k_{ad,2}$ and Lemma 5.17. The proof is complete. □

In what follows, we make use the following concept.

Definition 5.1. We say that a sequence of pairs

$$\left\{(A_k,y_k)=(A^{sym}_k+A^{skew}_k,y_k)\in L^1(\Omega;;\mathbb{M}^N)\times W_{A^{sym}_k}(\Omega)\right\}_{k\in\mathbb{N}}$$

τ-converges to a pair $(A,y)=(A^{sym}+A^{skew},y)\in L^1(\Omega;;\mathbb{M}^N)\times W_{A^{sym}}(\Omega)$ if

$$\begin{aligned}&A^{sym}_k\to A^{sym}\ \text{ in }\ L^1(\Omega;\mathbb{S}^N_{sym}),\quad A^{skew}_k\to A^{skew}\ \text{ in }\ L^{2p}(\Omega;\mathbb{S}^N_{skew}),\\ &y_k\rightharpoonup y\quad \text{ in }\ L^2(\Omega),\quad \nabla y_k\rightharpoonup\nabla y\ \text{ in variable space }\ L^2(\Omega,A^{sym}_k\,dx)^N.\end{aligned}$$

We are now in a position to study the optimal control problems (5.11).

Theorem 5.1. *Let $f\in L^{4p/(p+1)}(\Omega;\mathbb{R}^N)$ and $y_d\in L^2(\Omega)$ be given distributions. Assume that the original OCP (4.3)–(4.4) has a nonempty set of admissible controls. Then OCPs (5.11) are regular for each $k\in\mathbb{N}$) (i.e. the corresponding sets of admissible solutions Ξ_k are nonempty), and for every $k\in\mathbb{N}$ there exists a minimizer $(A^0_k,y^0_k)\in\Xi_k$ to the corresponding minimization problems (5.11) such that the sequence of pairs $\left\{(A^0_k,y^0_k)\in\Xi_k\right\}_{k\in\mathbb{N}}$ is relatively compact with respect to the τ-convergence and each of its τ-cluster pairs $(\widehat{A},\widehat{y})$ possesses the properties:*

$$(\widehat{A},\widehat{y})\in\Xi, \tag{5.19}$$

$$\int_\Omega\left(\nabla\widehat{y},\widehat{A}^{sym}\nabla\widehat{y}\right)_{\mathbb{R}^N}dx+\int_\Omega a_0(\widehat{y})^2\,dx\le\int_\Omega(f,\nabla\widehat{y})_{\mathbb{R}^N}\,dx. \tag{5.20}$$

Proof. Since $\mathfrak{A}_{ad} \neq \emptyset$, it follows that $\mathfrak{A}_{ad}^k \neq \emptyset$ for every $k \in \mathbb{N}$, and $\mathfrak{A}_{ad}^k \subset L^\infty(\Omega;\mathbb{M}^N)$. Hence, for any admissible control $A_k = A_k^{sym} + A_k^{skew} \in \mathfrak{A}_{ad}^k$, we can claim that $A_k^{skew} \in L^\infty(\Omega;\mathbb{S}^N_{skew})$, $A_k^{sym} \in L^\infty(\Omega;\mathbb{S}^N_{sym})$, and, therefore, the corresponding bilinear form

$$\begin{aligned}
[y,\varphi]_{A_k} &= \int_\Omega \left(\nabla\varphi, A_k^{skew}\nabla y\right)_{\mathbb{R}^N} dx \\
&= \int_\Omega \Big(\left(A_k^{sym}\right)^{1/2}\nabla\varphi, \underbrace{\left[\left(A_k^{sym}\right)^{-1/2} A_k^{skew} \left(A_k^{sym}\right)^{-1/2}\right]}_{C_k} \left(A_k^{sym}\right)^{1/2}\nabla y\Big)_{\mathbb{R}^N} dx \\
&\le \|C_k\|_{L^\infty(\Omega;\mathbb{S}^N_{skew})}\|y\|_{A_k^{sym}}\|\varphi\|_{A_k^{sym}} \text{ is bounded on } W_{A_k^{sym}}(\Omega)
\end{aligned}$$

and satisfies the identity $\int_\Omega \left(\nabla\varphi, A_k^{skew}\nabla y\right)_{\mathbb{R}^N} dx = -\int_\Omega \left(\nabla y, A_k^{skew}\nabla\varphi\right)_{\mathbb{R}^N} dx$. Therefore,

$$\int_\Omega \left(\nabla v, A_k^{skew}(x)\nabla v\right)_{\mathbb{R}^N} dx = 0 \quad \forall\, v \in W_{A_k^{sym}}(\Omega) \tag{5.21}$$

and, hence, boundary value problem (5.13) has a unique solution $y_k \in W_{A_k^{sym}}(\Omega)$ for each $A_k \in \mathfrak{A}_{ad}^k \subset L^\infty(\Omega;\mathbb{M}^N)$ by the Lax-Milgram lemma. Thus, $\Xi_k \neq \emptyset$ for every $k \in \mathbb{N}$. It is worth to note that in view of the definition of the class of admissible controls $\mathfrak{A}_{ad}^k$ (see Lemma 5.4), the norms $\|\cdot\|_{H_0^1(\Omega}$ and $\|\cdot\|_{A_k^{sym}}$ are equivalent, therefore, we can identify $H_0^1(\Omega$ with the weighted Sobolev space $W_{A_k^{sym}}(\Omega)$.

As obvious consequence of this observation and the property of τ-lower semicontinuity of the cost functional I_k, we conclude that the corresponding minimization problem (5.11) admits at least one solution $(A_k^0, y_k^0) \in \Xi_k$ [15]. Moreover, having fixed a control $A_k \in \mathfrak{A}_{ad}^k$, condition (5.21) implies the fulfilment of the following identities for every $k \in \mathbb{N}$

$$\int_\Omega \left[\left(\nabla\varphi, A_k\nabla y_k\right)_{\mathbb{R}^N} + a_0\varphi y_k\right]\, dx = \int_\Omega (f,\nabla\varphi)_{\mathbb{R}^N}\, dx,\ \forall\,\varphi \in C_0^\infty(\Omega), \tag{5.22}$$

$$\int_\Omega \left(\nabla y_k, A_k^{sym}\nabla y_k\right)_{\mathbb{R}^N}\, dx + \int_\Omega a_0 y_k^2\, dx = \int_\Omega (f,\nabla y_k)_{\mathbb{R}^N}\, dx, \tag{5.23}$$

where $y_k = y_k(A_k, f) \in W_{A_k^{sym}}(\Omega)$ are the corresponding solutions to the boundary value problems (5.13). Taking into account estimate (4.14), the equality (5.23) implies that

$$\begin{aligned}
\|y_k\|^2_{A_k^{sym}} &:= \int_\Omega \left[\left(\nabla y_k, A_k^{sym}\nabla y_k\right)_{\mathbb{R}^N} + y_k^2\right]\, dx \\
&\le \frac{1}{\min\left\{1, \|a_0\|_{L^\infty(\Omega)}\right\}}\int_\Omega \left[\left(\nabla y_k, A_k^{sym}\nabla y_k\right)_{\mathbb{R}^N} + a_0 y_k^2\right]\, dx \\
&\le \frac{\sqrt{\|\zeta_k^{-1}\|_{L^{2q}(\Omega)}}}{\min\left\{1, \|a_0\|_{L^\infty(\Omega)}\right\}}\|f\|_{L^{4p/(p+1)}(\Omega;\mathbb{R})^N}\|y_k\|_{A_k^{sym}}.
\end{aligned}$$

Hence,

$$\sup_{k\in\mathbb{N}} \|y_k\|_{A_k^{sym}} \le \frac{\sqrt{\|\zeta_k^{-1}\|_{L^{2q}(\Omega)}}}{\min\left\{1, \|a_0\|_{L^\infty(\Omega)}\right\}} \|f\|_{L^{4p/p+1}(\Omega;\mathbb{R})^N} = C_1 \|f\|_{L^{4p/p+1}(\Omega;\mathbb{R})^N} \tag{5.24}$$

and, therefore, the sequence $\{y_k\}_{k\in\mathbb{N}}$ is bounded in variable space $W_{A_k^{sym}}(\Omega)$. As a result, we arrive at the relation

$$\begin{aligned} I_k(A_k^0, y_k^0) = \inf_{(A,y)\in\Xi_k} I_k(A,y) &\le I_k(A_k, y_k) \le 2\|y_d\|^2_{L^2(\Omega)} + 2\|y_k\|^2_{A_k^{sym}} \\ &\le 2\|y_d\|^2_{L^2(\Omega)} + C_1^2 \|f\|^2_{L^{4p/p+1}(\Omega;\mathbb{R}^N)} \le C \quad \forall\, k\in\mathbb{N}. \end{aligned} \tag{5.25}$$

Thus, the sequence of minimal values for the problems (5.11) is uniformly bounded,

$$\sup_{k\in\mathbb{N}} \inf_{(u,y)\in\Xi_k} I_k(u,y) \le C \quad \text{for some } \ C>0. \tag{5.26}$$

Hence, $\sup_{k\in\mathbb{N}} \|y_k^0\|^2_{A_k^{0,sym}} < +\infty$.

In the meantime, due to the definition of the sets $\mathfrak{A}_{ad}^k$, it is easy to see that the corresponding sequence of optimal controls $\left\{A_k^0\right\}_{k\in\mathbb{N}}$ belongs to $\mathfrak{A}_{ad,1}^k \oplus \mathfrak{A}_{ad,2}^k$. Hence, by Lemmas 5.5 and 5.7, we get: there exists a matrix $\widehat{A} \in \mathfrak{A}_{ad}^k$ such that

$$A_k^0 := A_k^{0,sym} + A_k^{0,skew} \to \widehat{A}^{sym} + \widehat{A}^{skew} =: \widehat{A} \ \text{ in } \ L^1(\Omega; \mathbb{M}^N), \tag{5.27}$$

$$A_k^{0,sym} \to \widehat{A}^{sym} \quad \text{in } \ L^1(\Omega; \mathbb{S}^N_{sym}), \tag{5.28}$$

$$A_k^{0,skew} \to \widehat{A}^{skew} \quad \text{in } \ L^{2p}(\Omega; \mathbb{S}^N_{skew}). \tag{5.29}$$

Therefore, taking into account Lemmas 5.6 and 5.8, we conclude: $\widehat{A} \in \mathfrak{A}_{ad}$.

Since $\sup_{k\in\mathbb{N}} \|y_k^0\|^2_{A_k^{0,sym}} < +\infty$, it follows by Lemma 4.1 that there exists an element $\widehat{y} \in W_{\widehat{A}^{sym}}(\Omega)$ such that, up to a subsequence, we have

$$y_k^0 \rightharpoonup \widehat{y} \ \text{ in } \ L^2(\Omega), \quad \nabla y_k^0 \rightharpoonup \nabla \widehat{y} \ \text{ in } \ L^2(\Omega, A_k^{0,sym}\, dx)^N. \tag{5.30}$$

As a result, summing up the above properties of the sequences $\left\{y_k^0\right\}_{k\in\mathbb{N}}$ and $\left\{A_k^0\right\}_{k\in\mathbb{N}}$, we obtain $(A_k^0, y_k^0) \xrightarrow{\tau} (\widehat{A}, \widehat{y})$.

The next step is to show that $(\widehat{A}, \widehat{y}) \in \Xi$. With that in mind, we pass to the limit in (5.22) with $A = A_k^0$ and $y = y_k^0$ as $k \to \infty$ using the properties (5.27)–(5.30). Having fixed a test function $\varphi \in C_0^\infty(\Omega)$, we get (see definition of the weak convergence in variable spaces)

$$\begin{aligned} \int_\Omega (f, \nabla\varphi)_{\mathbb{R}^N}\, dx &= \lim_{k\to\infty} \int_\Omega \left[\left(\nabla\varphi, A_k^0 \nabla y_k^0\right)_{\mathbb{R}^N} + a_0 \varphi y_k^0 \right] dx = \big(\text{by } (5.28),(5.30)\big) \\ &= \lim_{k\to\infty} \int_\Omega \left(\nabla\varphi, A_k^{0,skew} \nabla y_k^0\right)_{\mathbb{R}^N} dx + \int_\Omega \left(\nabla\varphi, \widehat{A}^{sym} \nabla\widehat{y}\right)_{\mathbb{R}^N} dx + \int_\Omega a_0 \varphi \widehat{y}\, dx. \end{aligned}$$

Since $\lim_{k\to\infty}\int_\Omega \left(\nabla\varphi, A_k^{0,skew}\nabla y_k^0\right)_{\mathbb{R}^N} dx = \lim_{k\to\infty} I_{1,k} + \lim_{k\to\infty} I_{2,k}$, where

$$
\begin{aligned}
I_{1,k} &= -\int_\Omega \Big(\left(A_k^{0,sym}\right)^{-\frac{1}{2}} (A_k^{0,skew} - \widehat{A}^{skew})\nabla\varphi, \left(A_k^{0,sym}\right)^{\frac{1}{2}} \nabla y_k^0\Big)_{\mathbb{R}^N} dx,\\
I_{2,k} &= -\int_\Omega \Big(\left(A_k^{0,sym}\right)^{-1} \widehat{A}^{skew}\nabla\varphi, A_k^{0,sym}\nabla y_k^0\Big)_{\mathbb{R}^N} dx,
\end{aligned}
$$

it follows from (4.30)–(4.31) that $\lim_{k\to\infty} I_{2,k} = \int_\Omega \left(\nabla\varphi, \widehat{A}^{skew}\nabla\widehat{y}\right)_{\mathbb{R}^N} dx$ and $\lim_{k\to\infty} I_{1,k} = 0$. Thus, the τ-limit pair $(\widehat{A}, \widehat{y}\,)$ is related by integral identity

$$
\int_\Omega \Big[\left(\nabla\varphi, \widehat{A}\nabla\widehat{y}\right)_{\mathbb{R}^N} + a_0\varphi\widehat{y}\Big]\, dx = \int_\Omega (f, \nabla\varphi)_{\mathbb{R}^N}\, dx,\ \forall\,\varphi \in C_0^\infty(\Omega),
$$

and, hence, $\widehat{y}$ is a weak solution to the boundary value problem (4.1)–(4.2) under $A = \widehat{A}$. Thus, $\widehat{y} \in D(\widehat{A})$ and, therefore, this pair is admissible for the original OCP (4.3)–(4.4), i.e. $(\widehat{A}, \widehat{y}\,) \in \Xi$.

It remains to prove the energy inequality (5.20). To this end, we pass to the limit in the energy equality (5.23) using the lower semicontinuity of the norms $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^2(\Omega; A_k^{0,sym}\,dx)^N}$ with respect to the weak convergence (5.30). As a result, we have

$$
\begin{aligned}
\int_\Omega (f, \nabla\widehat{y})_{\mathbb{R}^N}\, dx &= \lim_{k\to\infty}\int_\Omega \left(f, \nabla y_k^0\right)_{\mathbb{R}^N}\, dx = \lim_{k\to\infty}\int_\Omega \left(\nabla y_k^0, A_k^{0,sym}\nabla y_k^0\right)_{\mathbb{R}^N} dx\\
&+ \lim_{k\to\infty}\int_\Omega a_0 (y_k^0)^2\, dx \overset{\text{by Proposition 3.2}}{\geq} \int_\Omega \left(\nabla\widehat{y}, \widehat{A}^{sym}\nabla\widehat{y}\right)_{\mathbb{R}^N} dx + \int_\Omega a_0\widehat{y}^2\, dx
\end{aligned}
\tag{5.31}
$$

The proof is complete. □

Remark 5.2. As energy inequality (5.20) indicates, if $(\widehat{A}, \widehat{y}\,) \in \Xi$ is a τ-cluster pair of the sequence $\left\{(A_k^0, y_k^0) \in \Xi_k\right\}_{k\in\mathbb{N}}$ and $\widehat{y} \in H_{\widehat{A}_{sym}}(\Omega)$, then the direct comparison of (5.20) and (4.16) implies that $[\widehat{y}, \widehat{y}\,]_{\widehat{A}} \geq 0$.

As immediately follows from this theorem, Hypothesis A can be eliminated from Theorem 4.2. Namely, we have the following result.

Corollary 5.1. *If $\mathfrak{A}_{ad} \neq \emptyset$, then the set of admissible solutions Ξ to OCP* (4.3)–(4.4) *is nonempty for every $f \in L^{4p/(p+1)}(\Omega; \mathbb{R}^N)$ and $y_d \in L^2(\Omega)$.*

Remark 5.3. As follows from Theorem 5.1, for any positive compactly supported smooth function K satisfying conditions (5.3), optimal solutions to the regularized OCPs (5.11) always lead in the limit to some admissible (but not optimal in general) solution $(\widehat{A}, \widehat{y}\,)$ of the original OCP (4.3)–(4.4). Moreover, in general, this limit pair depends on the choice of smoothing kernel K. It is reasonably to call such pair attainable. However, up to now the structure of the entire set of all attainable pairs remains unclear. For instance, it is unknown whether this set is convex and closed in Ξ. It is also unknown whether all optimal solutions to OCP (4.3)–(4.4) can be attainable in such way.

Taking these observations into account, we make use of the following notion.

Definition 5.2. We say that a pair $(\widehat{A},\widehat{y}\,)\in L^1(\Omega;\mathbb{M}^N)\times W_{\widehat{A}_{sym}}(\Omega)$ is a variational solution to OCP (4.3)–(4.4) if

$$I(\widehat{A},\widehat{y}\,)=\inf_{(A,y)\in\Xi}I(A,y),\quad (\widehat{A},\widehat{y}\,)\in\Xi, \tag{5.32}$$

and $(\widehat{A},\widehat{y}\,)$ is related by energy equality

$$\int_\Omega\left(\nabla\widehat{y},\widehat{A}^{sym}\nabla\widehat{y}\right)_{\mathbb{R}^N}\,dx+\int_\Omega a_0(\widehat{y}\,)^2\,dx=\int_\Omega(f,\nabla\widehat{y})_{\mathbb{R}^N}\,dx. \tag{5.33}$$

As a consequence of Theorem 5.1 and properties of the variational limits of constrained minimization problems (see Theorem 2.1), we have the following result.

Proposition 5.1. Let K be a smoothing kernel with properties (5.3). Assume that the sequence of minimization problems defined by the rules (5.12)–(5.13) is such that

$$\left\langle\inf_{(A,y)\in\,\Xi_k}I_k(A,y)\right\rangle\xrightarrow[k\to\infty]{\mathrm{Var}(\tau)}\left\langle\inf_{(A,y)\in\Xi}I(A,y)\right\rangle\quad\text{(see Definition 2.3).} \tag{5.34}$$

Let $\left\{(A_k^0,y_k^0)\in\Xi_k\right\}_{k\in\mathbb{N}}$ be a sequence of optimal solutions to the corresponding regularized OCPs. Then this sequence is relatively compact with respect to the τ-convergence and each its τ-cluster pair $(\widehat{A},\widehat{y}\,)\in L^1(\Omega;\mathbb{M}^N)\times W_{\widehat{A}_{sym}}(\Omega)$ is a variational solution to OCP (4.3)–(4.4) in the sense of Definition 5.2. Moreover, up to a subsequence, we have

$$y_k^0\to\widehat{y}\ \text{ in }\ L^2(\Omega)\ \text{ and }\ \nabla y_k^0\to\nabla\widehat{y}\ \text{ in }\ L^2(\Omega,A_k^{0,sym}\,dx)^N\ \text{ as } k\to\infty. \tag{5.35}$$

Proof. Indeed, the τ-compactness of the sequence $\left\{(A_k^0,y_k^0)\in\Xi_k\right\}_{k\in\mathbb{N}}$ is a direct consequence of a priori estimate (5.24), Lemma 4.1, and properties (5.27)–(5.29). In order to prove the the strong convergence (5.35), we make use of the main properties of the variational convergence. Following Theorems 2.1, 5.1, and 4.2 (see also Corollary 5.1), we can claim that OCP (4.3)–(4.4) is solvable and there exists an optimal pair $(A^0,y^0)\in\Xi$ to this problem such that

$$\begin{aligned}\inf_{(A,y)\in\,\Xi}I(A,y)=&I\left(A^0,y^0\right):=\left\|y^0-y_d\right\|^2_{L^2(\Omega)}+\int_\Omega\left(\nabla y^0,A^{0,sym}\nabla y^0\right)_{\mathbb{R}^N}\,dx\\ &=\lim_{k\to\infty}\inf_{(A_k,y_k)\in\,\Xi_k}I_k(A_k,y_k)=\lim_{k\to\infty}I_k(A_k^0,y_k^0)\\ &=\lim_{k\to\infty}\left[\left\|y_k^0-y_d\right\|^2_{L^2(\Omega)}+\int_\Omega\left(\nabla y_k^0,A_k^{0,sym}\nabla y_k^0\right)_{\mathbb{R}^N}\,dx\right].\end{aligned} \tag{5.36}$$

However, because of the lower semicontinuity of $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^2(\Omega,A_k^{0,sym}\,dx)^N}$ with respect to the weak convergence, the convergence $(A_k^0,y_k^0)\xrightarrow{\tau}(A^0,y^0)$ implies that

$$\begin{aligned}\inf_{(A,y)\in\Xi} I(A,y) &\overset{\text{by }(5.36)}{=} \lim_{k\to\infty}\left[\left\|y_k^0-y_d\right\|^2_{L^2(\Omega)}+\int_\Omega\left(\nabla y_k^0, A_k^{0,sym}\nabla y_k^0\right)_{\mathbb{R}^N}\,dx\right]\\ &\geq \|\widehat{y}-y_d\|^2_{L^2(\Omega)}+\int_\Omega\left(\nabla\widehat{y},\widehat{A}^{sym}\nabla\widehat{y}\right)_{\mathbb{R}^N}\,dx.\end{aligned}$$

Since the pair $(\widehat{A},\widehat{y})$ is admissible for the problem (4.3)–(4.4) (see Theorem 5.1), it follows that $(\widehat{A},\widehat{y})$ is an optimal pair. Therefore, in view of (5.36), it gives

$$\begin{aligned}\inf_{(A,y)\in\Xi} I(A,y) =& I\left(\widehat{A},\widehat{y}\right) := \|\widehat{y}-y_d\|^2_{L^2(\Omega)}+\int_\Omega\left(\nabla\widehat{y},\widehat{A}^{sym}\nabla\widehat{y}\right)_{\mathbb{R}^N}\,dx\\ &= \lim_{k\to\infty}\inf_{(A_k,y_k)\in\Xi_k} I_k(A_k,y_k)=\lim_{k\to\infty} I_k(A_k^0,y_k^0)\\ &= \lim_{k\to\infty}\left[\left\|y_k^0-y_d\right\|^2_{L^2(\Omega)}+\int_\Omega\left(\nabla y_k^0, A_k^{0,sym}\nabla y_k^0\right)_{\mathbb{R}^N}\,dx\right].\end{aligned}\tag{5.37}$$

Hence, the validity of (5.35) is a direct consequence of properties (5.36)–(5.37) and Proposition 3.3. It remains to prove the energy equality (5.33). To this end, it is enough to note that each of the pair (A_k^0,y_k^0) is related by energy equality (5.23). As a result, passing to the limit in (5.23) as $k\to\infty$, we finally have

$$\begin{aligned}0 &\overset{\text{by }(5.21)}{=} \lim_{k\to\infty}[y_k^0,y_k^0]_{A_k^0} \overset{\text{by }(5.23)}{=} -\lim_{k\to\infty}\int_\Omega\left(\nabla y_k^0, A_k^{0,sym}\nabla y_k^0\right)_{\mathbb{R}^N}\,dx\\ &\quad -\lim_{k\to\infty}\int_\Omega a_0(y_k^0)^2\,dx+\lim_{k\to\infty}\int_\Omega (f,\nabla y_k^0)_{\mathbb{R}^N}\,dx\\ &\overset{\text{by }(5.35)\text{ and }(5.37)}{=} -\int_\Omega\left(\nabla\widehat{y},\widehat{A}^{sym}\nabla\widehat{y}\right)_{\mathbb{R}^N}\,dx-\int_\Omega a_0(\widehat{y})^2\,dx+\int_\Omega(f,\nabla\widehat{y})_{\mathbb{R}^N}\,dx.\end{aligned}$$

□

Remark 5.4. As follows from Proposition 5.1 and Theorem 5.1, even if the OCP (4.3)–(4.4) has a unique solution (A^0,y^0), it does not ensure that this pair is the variational solution to the above problem. The matter is that the existence at least one the smoothing kernel K such that the approximated OCPs (5.11)–(5.13) would lead to the pair (A^0,y^0) in the sense of conditions (2.25)–(2.26) is an open problem. In other words, the existence of (Γ,δ)-realizing sequence for the pair $(A^0,y^0)\in\Xi$ (see Definition 2.3) is not established.

We are now in a position to discuss the existence of variational solutions to the OCP (4.3)–(4.4).

Theorem 5.2. *Assume that*

- *the function ζ_{ad} in definition of the set $\mathfrak{M}^\beta_{\zeta_{ad}}(\Omega)$ satisfies conditions $\zeta_{ad}^{-1}\in L^{2q}(\Omega)$ with $q=p/(p-1)$, where p is defined by (2.15);*

- *condition* (4.22) *holds true for some constant* $C > 0$*;*
- *for every admissible control* $A \in \mathfrak{A}_{ad} \subset L^1\big(\Omega;\mathbb{M}^N\big)$*, we have*

$$[y, y]_A = 0 \quad \forall\, y \in D(H_{A_{sym}}). \tag{5.38}$$

Then the OCP (4.3)–(4.4) *has variational solutions for every* $f \in L^{4p/(p+1)}(\Omega;\mathbb{R}^N)$ *and* $y_d \in L^2(\Omega)$*.*

Proof. To begin with, we note that as follows from Theorem 4.1, condition (4.22) guarantees the fulfilment of equality $H_{A_{sym}}(\Omega) = W_{A_{sym}}(\Omega)$ for every admissible control $A = A_{sym} + A_{skew} \in \mathfrak{A}_{ad}$. Moreover, in this case, every weak solution to the boundary value problem (4.1)–(4.2) satisfies the energy equality (4.16). Let K be aa arbitrary positive compactly supported smooth function satisfying conditions (5.3). We associate with this function the sequence of constrained minimization problems (5.11), where the cost functional I_k and the set Ξ_k are defined by (5.12)–(5.13).

Let $\{(A_k, y_k)\}_{k\in\mathbb{N}}$ be a sequence in $L^1(\Omega;\mathbb{M}^N) \times W_{A_k^{sym}}(\Omega)$ with the following properties:

(a) $(A_k, y_k) \in \Xi_{n_k}$ for every $k \in \mathbb{N}$, where $\{n_k\}_{k\in\mathbb{N}}$ is a subsequence converging to ∞ as k tends to ∞;

(aa) $(A_k, y_k) \stackrel{\tau}{\to} (A, y)$ in the sense of Definition 5.1.

Then proceeding as in the proof of Theorem 5.1, it can be shown that the limit pair (A, y) is admissible to the original OCP (4.3)–(4.4). Hence, this problem is regular and, therefore, it is solvable by Theorem 4.2. Our aim is to show that this problem can be interpreted as the variational limit of the sequence of constrained minimization problems (5.11). To do so, we have to verify the fulfilment of all conditions of Definition 2.3.

As for the property (d), it immediately follows from the following relation

$$\begin{aligned}\liminf_{k\to\infty} I_k(A_k, y_k) = &\liminf_{k\to\infty} \left[\|y_k - y_d\|^2_{L^2(\Omega)} + \int_\Omega \big(\nabla y_k, A_k^{sym} \nabla y_k\big)_{\mathbb{R}^N}\, dx\right]\\ &\stackrel{\text{by (3.5)}}{\geq} \|y - y_d\|^2_{L^2(\Omega)} + \int_\Omega (\nabla y, A^{sym}\nabla y)_{\mathbb{R}^N}\, dx = I(A, y),\end{aligned}$$

which holds true for any sequence $\Big\{(A_k, y_k) \in \mathfrak{A}_{ad} \times W_{A_k^{sym}}(\Omega)\Big\}_{k\in\mathbb{N}}$ with properties (a)–(aa).

We focus now on the verification of condition (dd) of Definition 2.3. Let $(A^\sharp, y^\sharp)$ be an arbitrary admissible pair to the original problem. Since $A^\sharp \in \mathfrak{A}_{ad}$, it follows from Lemmas 5.6 and 5.8 that the sequence of smoothed matrices

$$\left\{A_k^\sharp := (A^{sharp})_k := \int_{\mathbb{R}^N} K\,(z)\, \widetilde{A}^\sharp(x + k^{-1}z)\, dz\right\}_{k\in\mathbb{N}}$$

such that $A_k^\sharp \in \mathfrak{A}_{ad}^k$ for all $k \in \mathbb{N}$, and $A_k^\sharp \to A^\sharp$ as $k \to \infty$ in the sense of Definition 5.1, i.e.

$$A_k^\sharp := A_k^{\sharp,sym} + A_k^{\sharp,skew} \to A^{\sharp,sym} + A^{\sharp,skew} =: A^\sharp \quad \text{in} \quad L^1(\Omega;\mathbb{M}^N), \tag{5.39}$$

$$A_k^{\sharp,sym} \to A^{\sharp,sym} \quad \text{in} \quad L^1(\Omega;\mathbb{S}^N_{sym}), \tag{5.40}$$

$$A_k^{\sharp,skew} \to A^{\sharp,skew} \quad \text{in} \quad L^{2p}(\Omega;\mathbb{S}^N_{skew}). \tag{5.41}$$

Let $\left\{y_k = y(A_k^\sharp, f)\right\}_{k\in\mathbb{N}}$ be the corresponding solutions to the regularized boundary value problems (5.11). Then having applied the arguments of the proof of Theorem 5.1, it can be shown that the sequence $\{y_k\}_{k\in\mathbb{N}}$ is uniformly bounded in variable Sobolev space $W_{A_k^{\sharp,sym}}(\Omega)$ and there exists an element $\widehat{y} \in W_{A^{\sharp,sym}}(\Omega)$ such that $\widehat{y} \in D(W_{A^{\sharp,sym}})$, $(A^\sharp, \widehat{y}) \in \Xi$, and, up to a subsequence,

$$y_k \rightharpoonup \widehat{y} \quad \text{in} \quad L^2(\Omega), \quad \nabla y_k \rightharpoonup \nabla\widehat{y} \quad \text{in} \quad L^2(\Omega, A_k^{\sharp,sym}\,dx)^N. \tag{5.42}$$

Our aim is to show that $\widehat{y} = y^\sharp$ and the following identity

$$I(A^\sharp, y^\sharp) = \limsup_{k\to\infty} I_k(A_k^\sharp, y_k) \tag{5.43}$$

holds true.

Indeed, since $(A^\sharp, y^\sharp) \in \Xi$ and $(A^\sharp, \widehat{y}) \in \Xi$, it follows that $y = y^\sharp - \widehat{y}$ is a solution of the homogeneous problem

$$-\operatorname{div}\big(A\nabla y\big) + a_0 y = 0 \quad \text{in} \quad \Omega, \quad y = 0 \text{ on } \partial\Omega. \tag{5.44}$$

Following the initial assumptions, we have $W_{A^{sym}}(\Omega) = H_{A^{sym}}(\Omega)$ and $[y,y]_A = 0$ $\forall\, y \in D(W_{A^{sym}})$ and for each matrix $A \in \mathfrak{A}_{ad}$. Hence,

$$0 \overset{\text{by (5.38)}}{=} -[y,y]_{A^\sharp} \overset{\text{by (4.16)}}{=} \int_\Omega \left[\big(\nabla y, A^{\sharp,sym}\nabla y\big)_{\mathbb{R}^N} + a_0 y^2\right] dx$$

and, therefore, problem (5.44) has the trivial solution only. Thus, $y^\sharp = \widehat{y}$.

To prove the equality (5.43), we use of the idea of D.Cioranescu and F.Murat (see [2]). In view of the initial assumptions and Remark 2.1, the embedding $H_{A^{\sharp,sym}}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Taking into account this fact, the property (5.39), and the energy equalities (5.23) and (4.16), we get

$$\begin{aligned}
\lim_{k\to\infty} I_k(A_k^\sharp, y_k) &= \lim_{k\to\infty}\left[\|y_k - y_d\|^2_{L^2(\Omega)} + \int_\Omega \Big(\nabla y_k, A_k^{\sharp,sym}\nabla y_k\Big)_{\mathbb{R}^N}\,dx\right] \\
&= \Big\|y^\sharp - y_d\Big\|^2_{L^2(\Omega)} + \lim_{k\to\infty}\int_\Omega \Big(\nabla y_k, A_k^{\sharp,sym}\nabla y_k\Big)_{\mathbb{R}^N}\,dx \\
&\overset{\text{by (5.23)}}{=} \Big\|y^\sharp - y_d\Big\|^2_{L^2(\Omega)} + \lim_{k\to\infty}\left[-\int_\Omega a_0 y_k^2\,dx + \int_\Omega (f, \nabla y_k)_{\mathbb{R}^N}\,dx\right] \\
&\overset{\text{by (5.42)}}{=} \Big\|y^\sharp - y_d\Big\|^2_{L^2(\Omega)} + \int_\Omega \Big(f, \nabla y^\sharp\Big)_{\mathbb{R}^N}\,dx - [y^\sharp, y^\sharp]_{A^\sharp} \\
&\overset{\text{by (4.16)}}{=} \Big\|y^\sharp - y_d\Big\|^2_{L^2(\Omega)} + \int_\Omega \Big(\nabla y^\sharp, A^{\sharp,sym}\nabla y^\sharp\Big)_{\mathbb{R}^N}\,dx = I(A^\sharp, y^\sharp).
\end{aligned}$$

This concludes the proof. □

Our next observation shows that variational solutions do not exhaust the entire set of all possible solutions to the original OCP (4.3)–(4.4). With that in mind, we adopt the following concept.

Definition 5.3. We say that a pair $(A_0, y_0) \in \Xi$ is a non-variational solution to OCP (4.3)–(4.4) if

$$I(A_0, y_0) = \inf_{(A,y)\in\Xi} I(A, y), \quad (A_0, y_0) \in \Xi, \quad \text{and} \tag{5.45}$$

$$\int_\Omega (\nabla y_0, A_0^{sym} \nabla y_0)_{\mathbb{R}^N} \, dx + \int_\Omega a_0 y_0^2 \, dx \neq \int_\Omega (f, \nabla y_0)_{\mathbb{R}^N} \, dx. \tag{5.46}$$

Lemma 5.9. *Assume that there exists a matrix* $A_0 \in \mathfrak{A}_{ad}$ *and an element* $v \in D(H_{A_0^{sym}}) \subset H_{A_0^{sym}}(\Omega)$ *with property* $[v, v]_{A_0} \neq 0$*. Then there are distributions* $f \in \mathcal{D}'(\Omega; \mathbb{R}^N)$ *and* $y_d \in L^2(\Omega)$ *such that the optimal control problem*

$$\textit{Minimize } I(A, y) = \|y - y_d\|^2_{L^2(\Omega)} + \int_\Omega (\nabla y - \nabla y_d, A^{sym}(\nabla y - \nabla y_d))_{\mathbb{R}^N} \, dx \tag{5.47}$$

$$\textit{subject to the constraints } (4.1)\text{–}(4.2) \textit{ and } A \in \mathfrak{A}_{ad} \subset L^1(\Omega; \mathbb{M}^N) \tag{5.48}$$

has a non-variational solution in the sense of Definition 5.3.

Proof. We consider the OCP (5.47)–(5.48) with

$$y_d = v \quad \text{and} \quad f = -A_0 \nabla v. \tag{5.49}$$

Since $v \in D(H_{A_0^{sym}})$, it follows that $y_d \in L^2(\Omega)$, $\nabla v \in L^2(\Omega, A_0^{sym}\, dx)^N$, and, therefore, $(A_0^{sym})^{-1/2} f \in L^2(\Omega)^N$. Indeed, as follows from $(5.49)_2$, we have $(A_0^{sym})^{-1/2} f = f_1 + f_2$, where $f_1 = (A_0^{sym})^{1/2} \nabla v$ and $f_2 = (A_0^{sym})^{-1/2} A_0^{skew} \nabla v$. Then

$$\int_\Omega |f_1|^2 \, dx = \int_\Omega (\nabla v, A_0^{sym} \nabla v)_{\mathbb{R}^N} \, dx \overset{v\in H_{A_0^{sym}}(\Omega)}{<} +\infty,$$

$$\int_\Omega |f_2|^2 \, dx = \int_\Omega | \underbrace{(A_0^{sym})^{-1/2} A_0^{skew} (A_0^{sym})^{-1/2}}_{C(x)} (A_0^{sym})^{1/2} \nabla v| \, dx$$

$$\leq \|C\|_{L^2(\Omega; \mathbb{S}^N_{skew})} \|v\|_{L^2(\Omega, A_0^{sym}\, dx)^N} \overset{\text{by } (2.17)}{<} +\infty.$$

Hence, $(A_0^{sym})^{-1/2} f \in L^2(\Omega)^N$ and by Corollary 4.1 we conclude that y_d is a weak solution to the boundary value problem (4.1)–(4.2) under $A = A_0$. Since $v \in D(H_{A_0^{sym}}) \subset H_{A_0^{sym}}(\Omega)$, it follows that (see Remark 4.2) the distribution y_d satisfies the energy equality

$$\int_\Omega (\nabla y_d, A_0^{sym} \nabla y_d)_{\mathbb{R}^N} \, dx + \int_\Omega a_0 y_d^2 \, dx + [y_d, y_d]_{A_0} = \int_\Omega (f, \nabla y_d)_{\mathbb{R}^N} \, dx. \tag{5.50}$$

Moreover, using the fact that $I(A_0, y_d) = 0$, we finally conclude: (A_0, y_d) is the unique optimal pair to the above OCP.

Our aim is to show that (A_0, y_d) is a non-variational solution to this problem. To this end, we assume, for a moment, that (A_0, y_d) is a variational solution. Then Proposition 5.1 guarantees the validity of the related

$$\int_\Omega \left(\nabla y_d, A_0^{sym} \nabla y_d\right)_{\mathbb{R}^N} dx + \int_\Omega a_0 y_d^2\, dx = \int_\Omega (f, \nabla y_d)_{\mathbb{R}^N}\ dx.$$

On the other hand, since $[y_d, y_d]_{A_0} := [v, v]_{A_0} \neq 0$, energy equality (5.49) leads to the strict inequality

$$\int_\Omega \left(\nabla y_d, A_0^{sym} \nabla y_d\right)_{\mathbb{R}^N} dx + \int_\Omega a_0 y_d^2\, dx \neq \int_\Omega (f, \nabla y_d)_{\mathbb{R}^N}\ dx$$

and, hence, we arrive at the contradiction with the previous assertion. Thus, (A_0, y_d) is a non-variational solution to the above problem. The proof is complete. □

Remark 5.5. As follows from Theorem5.1, if $(A_0, y_0) \in \mathfrak{A}_{ad} \times H_{A_0^{sym}}(\Omega)$ is a non-variational solution such that $[y_0, y_0]_{A_0} < 0$, then this solutions can not be attainable through the limit of optimal solutions to the regularized problems (5.11)–(5.13).

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