Non-genericity of initial data with punctual [omega]-limit set

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Abstract. We show the existence of a nonlinear differential equation of gradient type for which the set of convergence tu a point is non generic

1. Introduction

Let us consider a function $f : \mathbb{R}^N \to \mathbb{R}$ of class C^∞ (the regularity is not our concern for the moment being). It defines a diffential equation of gradient type:

$$\begin{cases} x'(t) = -\nabla f(x(t)) \\ x(0) = x_0. \end{cases} \tag{1}$$

Obviously the map $t \mapsto f(x(t))$ is decreasing for a solution of (1). It is also well known that is we define

$$\omega(x_0) := \bigcap_{t \geq 0} \overline{\{x(s),\, s \geq t\}}, \tag{2}$$

then each set $\omega(x_0)$ is connected and compact when x is bounded. This is the case, for example, when f is proper.

For years and still now, the structure of the omega-limit set $\omega(x_0)$ has been widely studied and the properties of its equivalent couterpart for system of infinite dimension is a very active aera of interests.

A natural question that arises concerning $\omega(x_0)$ is wether it is a mere single point (of course an equilibrium point, *i.e. a critical point of* f). In this case we say that the trajectory converges. It is obviously the case if $\nabla f^{-1}(\{0\})$ is discrete. In two fondamentals papers [4] and [5], S. Lojasiewicz has proved that it is also the case when f is real analytic.

In the seminal book of J. Palis and W. De Melo ([6]), one can find an exemple of function $f : \mathbb{R}^2 \to \mathbb{R}$ such that for some x_0, $\omega(x_0)$ is not a single point (see also [1]). Of course such a f cannot be real analytic on the whole space.

In this paper we prove the following: Let $\mathcal{S}$ be the circle of center 0 and radius 1 in $\mathbb{R}^2$.

Theorem 1.1. *There exists a C^∞ function $f : \mathbb{R}^2 \to \mathbb{R}$ such that there exists an open set $\mathcal{O}$ satisfying:*

$$\forall x_0 \in \mathcal{O}, \quad \omega(x_0) = \mathcal{S}.$$

Remark 1. Let Ω be a bounded regular open set of $\mathbb{R}^N$. For $g : \mathbb{R} \to \mathbb{R}$, we introduce the semilinear heat equation

$$\begin{cases} \partial_t u - \Delta u = g(u), \text{ on } (0, \infty) \times \Omega \\ u = u_0 \text{ on } \{0\} \times \Omega \\ u = 0 \text{ on } (0, \infty) \times \partial\Omega. \end{cases} \tag{3}$$

We consider the functional on $H_0^1(\Omega)$

$$\Gamma(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega G(u) dx,$$

where G satisfies $G' = g$.
It is standard that (3) can be viewed as as gradient system on $H_0^1(\Omega)$

$$u' = -\nabla\Gamma(u).$$

Several results about convergence for the equation (1) have been extended to (3) with specific assumptions on g which we do not detail here.

The example of the book of J. Palis and W. De Melo ([6]) has been used in a penetrating analysis by P. Polacik and K. Rybakowski [7] (see also [8]) to extend the result to a nonlinear heat equation (see (3)).

In [3], P.L. Lions proved that for the semilinear equation , under some natural assumptions on g, there exists a dense open set of u_0 such that the solution converges to an equilibrium point. Our theorem proves that this is not the case in general for finite dimensional system.

Remark 2. In comparison to the example given in [7] whose construction relies on a connexity argument, our method relies on an explicit construction of surrounding curves, playing the role of sub and supersolutions.

Remark 3. In [2], it is remarked that the non-convergence of a solution of (1) implies the non-convergence of the solution of the second order differential equation

$$x'' + \alpha x' + \nabla\chi(x) = 0 \tag{4}$$

when one considers $\chi = \alpha f - \frac{1}{2}|\nabla f|^2$ with the following initial data $x(0) = x_0$ and $x'(0) = x_1 = -\nabla f(x_0)$.

Our result does not extend to the second order equation (4). In fact the set $\{(x_0, x_1)\}$ that we can construct with our method contains a submanifold of dimension 2 in $\mathbb{R}^4$. We skip the proof of this result. The existence of χ such that there exist an open set of initial data $\{(x_0, x_1)\}$ such that the solution to (4) does not converge to an equilibrium point is, to our knowledge, open.

2. Proof of the theorem 1.1

2.1. Construction of f

Let us consider two C^∞ curves defined in polar coordinate by two functions γ_i $i = 1,2$ such that

$$\forall \theta \in \mathbb{R}^+,\ 1 < \gamma_2(\theta + 2\pi) < \gamma_1(\theta) < \gamma_2(\theta) \tag{5}$$

$$\lim_{\theta\to\infty} \gamma_i(\theta) = 1. \tag{6}$$

Let us also consider an even C^∞ function

$$F : \mathbb{R} \longrightarrow [0,+\infty[,$$

such that the support of F is $[-1,1]$ and for some $\alpha \in (0,1)$

$$F(\alpha) = \max_{[-1,1]} F,$$

F is strictly increasing on $[0,\alpha]$ and F is positive on $]-1,1[$. An example of such function is given by

$$F(x) = \begin{cases} \frac{e^{-\frac{1}{\lambda(1+x)^2}-\frac{1}{\lambda(1-x)^2}}}{(1+x)^\nu(1-x)^\nu}, & \text{when } |x| < 1 \\ 0 \text{ else}, \end{cases}$$

for $\nu = 1,2$ and $\lambda > 0$ large enough with respect to ν.

Let us fix $\theta_0 \in (0,2\pi)$. We introduce a function Π such that its support is connected and contains $\gamma_1(0)$ and $\gamma_2(0)$ as interior points and such that for $\theta > \theta_0$ $\gamma_1(\theta)$ and $\gamma_2(\theta)$ are not in the support of Π.

Let us consider now some point $M := (x,y) \in \mathbb{R}^2$ such that $r := \sqrt{x^2+y^2} > 1$.

We choose some $\theta \in [0,2\pi)$ such that $M = (r\cos(\theta), r\sin(\theta))$.

If there does not exist $p \in \mathbb{N}$ such that $\gamma_1(\theta+2p\pi) \le r \le \gamma_2(\theta+2p\pi)$ we define $\phi(x,y) = 0$.

Otherwise, there exist a unique $p \in \mathbb{N}$ such that

$$\gamma_1(\theta+2p\pi) \le r \le \gamma_2(\theta+2p\pi).$$

In that case we consider

$$\phi(x,y) :=$$
$$F\left(\frac{r-\gamma_1(\theta+2p\pi)}{\gamma_2(\theta+2p\pi)-\gamma_1(\theta+2p\pi)} + \frac{r-\gamma_2(\theta+2p\pi)}{\gamma_2(\theta+2p\pi)-\gamma_1(\theta+2p\pi)}\right), \tag{7}$$

and thereafter define f by

$$f(x,y) := e^{-(\theta+2p\pi)^2}\phi(x,y)(1-\Pi(x,y)). \tag{8}$$

From now on, we will choose $\gamma_1(\theta) = 1+e^{-\theta}$ and $\gamma_2(\theta) = 1+\mu e^{-\theta}$ with $\mu > 1$, but we would like to point out that many other choices are possible.

With these functions we will define

$$X(x,y) := \nabla f(x,y).$$

In ordrer to prove our main result we need to check that $f \in C^\infty(\mathbb{R}^2)$. According to our choice of F and our construction, it is clear the $f \in C^\infty(\Omega_1 \cup \Omega_2)$ where $\Omega_1 = \{(x,y),\, x^2+y^2<1\}$ and $\Omega_2 = {}^c\bar{\Omega}_1$.

Let us prove that f is continuous at any (x,y) such that $x^2+y^2=1$

For that we fix such a $(\bar{x},\bar{y})$ and a sequence $(x_n,y_n)\to(\bar{x},\bar{y})$. We may of course assume that $f(x_n,y_n)\neq 0$.

Let us assume first that $(\bar{x},\bar{y}) = (\cos(\bar{\theta}),\sin(\bar{\theta}))$ for $\bar{\theta}\in(0,2\pi)$.

We can then assume that for some $\theta_n \in [0,2\pi)$ we have

$(x_n,y_n) = \rho_n(\cos(\theta_n),\sin(\theta_n))$ where $\rho_n > 1$ and thus up to a subsequence, we have either $\theta_n \to \bar{\theta}$ or $\bar{\theta}+2\pi$. Let $p_n \in \mathbb{N}$ be such that $\gamma_1(\theta_n+2p_n\pi) < \rho_n < \gamma_2(\theta_n+2p_n\pi)$. We claim that $p_n\to\infty$.

Indeed otherwise, up to a subsequence, we may assume that p_n is bounded and thus constant for n large enough and the sequence $(\theta_n+2p_n\pi)$ is thus bounded. But this contradicts the fact that $\rho_n\to 1$.

Since $\theta_n+2p_n\pi\to+\infty$, $f(x_n,y_n)\to 0$.

We have thus proven the continuity of f.

Let us prove that f is C^1.

Let us denote $h(x,y) = e^{-(\theta+2p\pi)^2}$.

We classicaly define

$$e_r(\theta) = (\cos(\theta),\sin(\theta)),\; e_\theta(\theta) = (-\sin(\theta),\cos(\theta)).$$

Then $X = \nabla f$ is given at (x,y) such that $f(x,y)\neq 0$.

$$\begin{aligned}
&X(x,y) = \\
&-\frac{2h(\theta+2p\pi)}{\gamma_2(\theta+2p\pi)-\gamma_1(\theta+2p\pi)}F'(a(r,\theta+2p\pi))e_r \\
&-\frac{h'(\theta+2p\pi)}{r}F(a(r,\theta+2p\pi))e_\theta \\
&-\frac{h(\theta+2p\pi)}{r}F'(a(r,\theta+2p\pi))\times \\
&\frac{1}{\gamma_2(\theta+2p\pi)-\gamma_1(\theta+2p\pi)}(-(\gamma_1'(\theta+2p\pi)+\gamma_2'(\theta+2p\pi)) \\
&-(2r-\gamma_1(\theta+2p\pi)-\gamma_2(\theta+2p\pi))\frac{\gamma_2'(\theta+2p\pi)-\gamma_1'(\theta+2p\pi)}{\gamma_2(\theta+2p\pi)-\gamma_1(\theta+2p\pi)})e_\theta,
\end{aligned}$$

where

$$a(r,\theta) := \frac{r-\gamma_1(\theta+2p\pi)}{\gamma_2(\theta+2p\pi)-\gamma_1(\theta+2p\pi)} + \frac{r-\gamma_2(\theta+2p\pi)}{\gamma_2(\theta+2p\pi)-\gamma_1(\theta+2p\pi),}$$

and p has been chose above.

If we compute $X(x_n,y_n)$ as before, since we will have $p_n\to\infty$ with our choice of h, $X(x_n,y_n)$ tends to 0. Thus the partial derivatives are continuous on $\mathbb{R}^2$.

Now a single argument by induction allows us to conclude that $f\in C^\infty(\mathbb{R}^2)$.

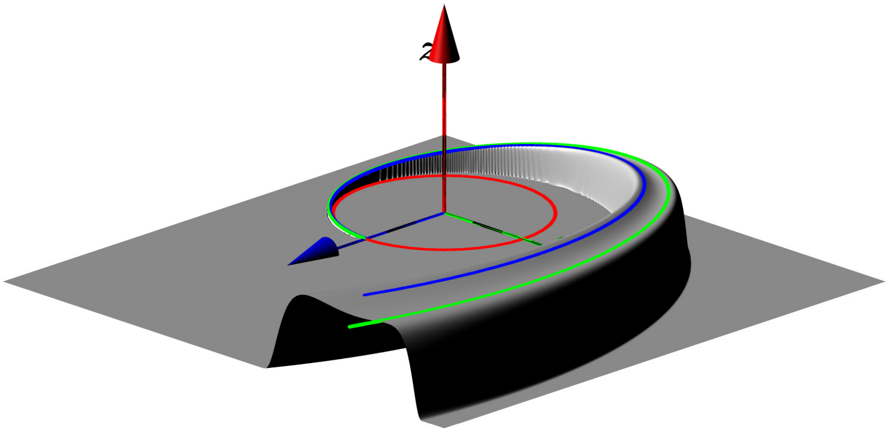

FIGURE 1. profile of f with $F(x,y) = \frac{e^{-\frac{1}{3(1+x)^2}-\frac{1}{3(1-x)^2}}}{(1+x)(1-x)}$. In red $\mathcal{S}$, and in blue and green the curves b_1 and b_2.

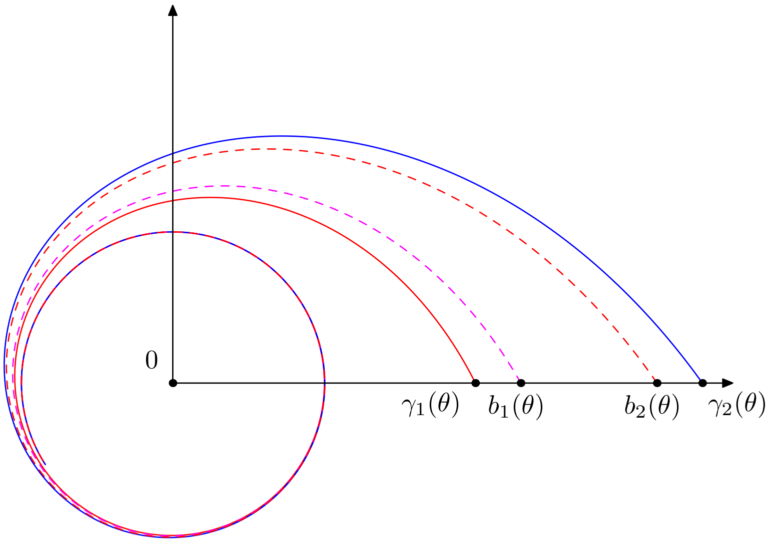

FIGURE 2. The domain on which f is not 0

2.2. f satisfies our problem

In this section we fix some (x,y) such that $f(x,y) \neq 0$. As in the previous section we define r and θ and p. However in order to simplify the exposition, since we will study the behavior of X along the half-line θ constant, we will replace $\theta + 2p\pi$ by θ, and what we have in mind is described in the pictures 1 and 2.

There are two values of r, $a_1(\theta)$ and $a_2(\theta)$, such for a given θ ϕ is maximum at a_1 and a_2. There are also two values of r, $b_1(\theta)$ and $b_2(\theta)$ such that

$$[b_1(\theta), b_2(\theta)] \subset]a_1(\theta), a_2(\theta)[,$$

and such that at those values $\gamma := |\partial_r \phi(r,\theta)|$ is maximum on $[a_1(\theta), a_2(\theta)]$. Of course

$$\partial_r \phi(b_1(\theta), \theta) < 0$$

and

$$\partial_r \phi(b_2(\theta), \theta) > 0.$$

These two values are given by

$$b_i(\theta) := \frac{\pm\beta(\gamma_2(\theta) - \gamma_1(\theta))}{2} + \frac{\gamma_1(\theta) + \gamma_2(\theta)}{2}$$

with $+$ when $i = 2$ and $-$ when $i = 1$ and $\pm\beta$ are the values in $[-\alpha, \alpha]$ where $|F'|$ is maximum on $[-\alpha, \alpha]$, while F is maximum in $-\alpha$ and α. We will denote $||F'||_\infty = \max\limits_{[-\alpha,\alpha]} |F'|$.

One can easily compute that

$$b_2(\theta) = \gamma_1(\theta)(1 - \frac{\beta}{2}) + \gamma_2(\theta)(1 + \frac{\beta}{2}),$$

and

$$b_1(\theta) = \gamma_1(\theta)(1 + \frac{\beta}{2}) + \gamma_2(\theta)(1 - \frac{\beta}{2}),$$

and since $\beta \in]-1, 1[$, the b_is are decreasing.
The tangent vector $\tau_i(\theta)$ to the curve $b_i(\theta)$ is given by

$$b_i'(\theta)e_r + b_i(\theta)e_\theta$$

The normal vector is then $n_i(\theta) = -b_i(\theta)e_r + b_i'(\theta)e_\theta$.
In order to prove our result, we show that for θ large enough one has

$$X(b_1(\theta), \theta)).n_1(\theta) < 0 \text{ and } X(b_2(\theta), \theta)).n_2(\theta) > 0.$$

Let us recall that

$$\begin{aligned} X(r,\theta) \quad = \quad & -\frac{2h(\theta)}{\gamma_2(\theta) - \gamma_1(\theta)} F'(a(r,\theta))e_r - \frac{h'(\theta)}{r} F(a(r,\theta))e_\theta \\ & -\frac{h(\theta)}{r} F'(a(r,\theta)) \frac{1}{\gamma_2(\theta) - \gamma_1(\theta)} (-(\gamma_1'(\theta) + \gamma_2'(\theta)) \\ & -(2r - \gamma_1(\theta) - \gamma_2(\theta)) \frac{\gamma_2'(\theta) - \gamma_1'(\theta)}{\gamma_2(\theta) - \gamma_1(\theta)}) e_\theta. \end{aligned}$$

Thus

$$\begin{aligned} & X(r,\theta).n_i(\theta) = 2\frac{h(\theta)}{\gamma_2(\theta) - \gamma_1(\theta)} F'(a(r,\theta)) b_i(\theta) \\ & -\frac{h'(\theta)}{r} F(a(r,\theta)) b_i'(\theta) - b_i'(\theta) \frac{h(\theta)}{r} \frac{F'(a(r,\theta))}{\gamma_2(\theta) - \gamma_1(\theta)} [-(\gamma_1'(\theta) + \gamma_2'(\theta)) \\ & -(2r - \gamma_1(\theta) - \gamma_2(\theta)) \frac{\gamma_2'(\theta) - \gamma_1'(\theta)}{\gamma_2(\theta) - \gamma_1(\theta)}] \end{aligned}$$

Now we take $r = b_1(\theta)$ and $i = 1$.
The preceding computation gives:

$$X(b_1(\theta),\theta).n_1(\theta) = -2\frac{h(\theta)}{\gamma_2(\theta)-\gamma_1(\theta)}||F'||_\infty b_1(\theta)$$
$$-\frac{h'(\theta)}{b_1(\theta)}F(a(b_1(\theta),\theta))b_1'(\theta)$$
$$+b_1'(\theta)\frac{h(\theta)}{b_1(\theta)}\frac{||F'||_\infty}{\gamma_2(\theta)-\gamma_1(\theta)}[-(\gamma_1'(\theta)+\gamma_2'(\theta))$$
$$-(2b_1(\theta)-\gamma_1(\theta)-\gamma_2(\theta))\frac{\gamma_2'(\theta)-\gamma_1'(\theta)}{\gamma_2(\theta)-\gamma_1(\theta)}].$$

Inside the bracket the quantity is

$$-(\gamma_1'(\theta)+\gamma_2'(\theta))+\beta(\gamma_2'(\theta)-\gamma_1'(\theta))$$

Let us recall that $\gamma_1(\theta) = 1 + e^{-\theta}$ and $\gamma_2(\theta) = 1 + \mu e^{-\theta}$, we thus get

$$X(b_1(\theta),\theta).n_1 = \tag{9}$$
$$-2\frac{h(\theta)}{(\mu-1)e^{-\theta}}||F'||_\infty(1+(\frac{1+\mu}{2}+\beta(\frac{1-\mu}{2}))e^{-\theta}) \tag{10}$$
$$+h'(\theta)F(a(b_1(\theta),\theta))\frac{\frac{\mu+1}{2}+\beta\frac{1-\mu}{2}}{1+e^{-\theta}(\frac{\mu+1}{2}+\beta\frac{1-\mu}{2})}e^{-\theta} \tag{11}$$
$$-\frac{h(\theta)||F'||_\infty(\frac{\mu+1}{2}+\beta\frac{1-\mu}{2})}{(1+e^{-\theta}(\frac{\mu+1}{2}+\beta\frac{1-\mu}{2}))(\mu-1)}((1+\beta)+\mu(1-\beta))e^{-\theta} \tag{12}$$

while

$$X(b_2(\theta),\theta).n_2(\theta) = 2\frac{h(\theta)}{\gamma_2(\theta)-\gamma_1(\theta)}||F'||_\infty b_2(\theta)$$
$$-\frac{h'(\theta)}{b_2(\theta)}F(a(r,\theta),\theta)b_2'(\theta)$$
$$-b_2'(\theta)\frac{h(\theta)}{b_2(\theta)}\frac{||F'||_\infty}{\gamma_2(\theta)-\gamma_1(\theta)}\left[-(\gamma_1'(\theta)+\gamma_2'(\theta))\right.$$
$$\left.-(2b_2(\theta)-\gamma_1(\theta)-\gamma_2(\theta))\frac{\gamma_2'(\theta)-\gamma_1'(\theta)}{\gamma_2(\theta)-\gamma_1(\theta)}\right]$$

which, with the previous choice of γ_i leads to

$$X(b_2(\theta),\theta).n_2(\theta) = 2\frac{h(\theta)}{\gamma_2(\theta)-\gamma_1(\theta)}||F'||_\infty b_2(\theta)$$
$$-\frac{h'(\theta)}{b_2(\theta)}F(a(r,\theta),\theta)b_2'(\theta)$$
$$-b_2'(\theta)\frac{h(\theta)}{b_2(\theta)}\frac{||F'||_\infty}{\gamma_2(\theta)-\gamma_1(\theta)}\left[(1+\mu)+\beta(\mu-1)\right]e^{-\theta}$$

Let us remark that in the latter equation the second term of the right hand side in neglectable with respect to the first as θ goes to infinity, while the third and the first are positive. Thus for θ large enough $X(b_2(\theta),\theta).n_2(\theta) > 0$.

In the former (the derivation of $X(b_1(\theta),\theta)$ the second and the third part of the right hand side is neglectable with respect to the first which is negative. Thus, if θ is large enough $X(b_1(\theta),\theta).n_1(\theta) < 0$.

We fix θ_0 so that for $\theta > \theta_0$

$$X(b_1(\theta),\theta).n_1(\theta) < 0 \text{ and } X(b_2(\theta),\theta).n_2(\theta) > 0.$$

We now take $\theta_1 > \theta_0$ so that $\sup f$ on $[a_1(\theta_1), a_2(\theta_1)] \cap \theta = \theta_1$ is less than $\inf f$ on $[a_1(\theta_0), a_2(\theta_0)] \cap \theta = \theta_2$. This is always possible due to our choice of h.

The two curves b_1 and b_2 are such that if $x_0 \in [b_1(\theta_1), b_2(\theta_1)]$ the solution $x' = \nabla X(x)$ which is x_0 at $t = 0$ does not cross b_1 nor b_2.
The function h having no critical point, we have proven our result.
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References

1. P.A. Absil, R. Mahony and B. Andrews, Convergence of the iterates of descent methods for analytic cost functions. SIAM J. Optim. **16** (2005), 531–547.
2. M.A. Jendoubi, P. Poláčik, Non-stabilizing solutions of semilinear hyperbolic and elliptic equations with damping. Proc. Roy. Soc. Edinburgh Sect. A **133** (2003), 1137–1153.
3. P.L. Lions, Structure of the set of the steady-state solutions and asymptotic behavior of semilinear heat equations, J. Differential Equations **53** (1984), 362–386.
4. S. Lojasiewicz, *Ensembles semi-analytiques*, Preprint, I.H.E.S. Bures-sur-Yvette, 1965.
5. S. Lojasiewicz, Une propriètè topologique des sous ensembles analytiques réels. Colloques internationaux du C.N.R.S #117. Les équations aux dérivées partielles (1963).
6. J. Palis and W. de Melo. *Geometric theory of dynamical systems.* Springer, New York, 1982.
7. P. Poláčik and K. Rybakowski, Nonconvergent bounded trajectories in semilinear heat equations. J. Differential Equations **124** (1996), 472–494.
8. P. Poláčik and F. Simondon, Nonconvergent bounded solutions of semilinear heat equations on arbitrary domains. J. Differential Equations **186** (2002), 586–610.

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