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Abstract

This paper deals with unknown input interval observer synthesis for discrete-time switched systems. First, a decomposition leading to obtain a subsystem not affected by the unknown input is presented. Second, an interval observer is designed based on the Input-to-State Stability (ISS). The gains are computed by solving Linear Matrix Inequalities (LMI) formulated based on multiple quadratic Lyapunov functions under average dwell time switching signals. In addition, a change of coordinates can be taken in order to ensure the positivity of the estimation errors. Finally, an explicit expression for the unknown input bounds is derived. Note that while the additive disturbances and the measurement noises are unknown but assumed to be bounded with known bounds, *the unknown input signals are neither bounded nor stochastic.*

Keywords: Interval Observer, Switched Systems, Average Dwell Time, Input-to-State Stability, Multiple Quadratic Lyapunov Functions

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1. Introduction

Unknown Input Observers (UIO) design has been widely investigated and frequently used in engineering implementation [37, 41]. One of the most well-known practical interests of such kind of observers is the fault detection and isolation problem [40]. UIO have been largely studied in several contexts: in particular some works are devoted to linear as well as bilinear systems [27, 16, 37, 30], nonlinear systems [1, 2], switched systems [18, 21] and so on. In recent years, interval observers design has received a growing attention in control theory and real-life applications due to the capacity of this cutting edge observer in estimating the transition of state variables of dynamical systems. In fact, they produce time-varying intervals in which the state variables are guaranteed to last all times while classical observers do not provide such property. The technique, which originates in [24], has been developed for several families of systems such as linear and bilinear [11, 15, 14, 32], nonlinear [34, 36, 22], switched systems [19, 31, 39, 43, 44]. The suggested approaches are mainly relied on combining change of coordinates with observer gain design methods to ensure both framer and stability properties of the estimation error. Another interval observer introduced in [38] is based on the T-N-L observer structure where T, N, L are design parameters. Compared with the classical method based on coordinate transformation, the interval observer proposed in [38] has some additional design degrees of freedom, which can be optimized to improve the estimation accuracy. In addition, interval observers-based controllers have been constructed in [33, 6].

None of the above works addressed the interval observation problem for systems subject to unknown inputs. As in the case of classical observers, unknown input often hampers and sometimes prohibits construction of interval observers. Some first results related to unknown input interval observer have been proposed in the literature, e.g., [10, 28, 9] for LTI and LPV systems. To the best of the authors' knowledge, the design of unknown input interval observers for switched systems has not been fully investigated in the literature and most of the proposed results (the readers can refer for instance to [20, 29]) have been developed for continuous-time systems. The discrete-time case has not been studied yet and is sufficiently different to deserve a separate treatment. This motivates the present work. Although some of the key ideas of the previous works may be used along this construction, it is worth pointing out that the estimators we propose are not derived directly because changing the system from continuous to discrete time not only raises changes of stability properties but also requires the estimation procedure of the unknown input to be properly adjusted. Moreover, the use of common Lyapunov function as in [20] to guarantee the stability is conservative and therefore it is hard that the set of LMI admits a solution. In the present paper, a multiple Lyapunov function is employed to relax this conservatism. The problem of optimizing the accuracy of the error between the upper and lower bounds, which has not been investigated in [20, 29], is also considered.

In this paper, the methodology proposed in [7] is used to decompose the state equation of the system into two subsystems by employing a nonsingular "disturbance-decoupling" state transformation [12]: the first one depends on the unknown input and, in the second one, the unknown input may be dropped. Next, another state transformation using a time invariant change of coordinates is performed in order to ensure the cooperativity property of the observation error so an interval observer can be designed in these new coordinates for the free-unknown input system. Then, one can deduce that lower and upper bounds for the state in the original basis. Finally, the estimation of the unknown input bounds is derived. The main contributions of this paper are

- The provided interval state estimate is insensitive against the presence of the (possibly unbounded) unmeasurable disturbance input. Furthermore, unknown inputs are not constrained to be a signal of any type (random or strategic) nor to follow any model. Thus, no prior 'useful' knowledge of the dynamics of unknown inputs is available. Therefore, they are suitable for representing adversarial attack signals [42].

- A novel interval observer for a class of discrete-time linear switched systems with unknown inputs is proposed and the LMI formulation is given to compute the gains based on the input-to-state stability (ISS) using a multiple Lyapunov function under an average dwell time. ISS is a useful stability notion for studying the robustness of control systems affected by exogenous inputs. Roughly speaking, a system is input-to-state stable if every state trajectory corresponding to a bounded control remains bounded, and the trajectory eventually becomes small if the input signals are small no matter what the initial states are. In the absence of exogenous inputs, an input-to-state stable system is globally asymptotically stable. The proposed scheme provides *simultaneously* stable estimation errors for the state and the unknown input in the sense of ISS.

The remainder of this paper is organized as follows. Some preliminaries are briefly presented in Section 2. In Section 3, a step-by-step interval observer design for discrete-time linear switched systems in the presence of the unknown input, additive disturbances and measurement noises is drawn. A numerical example is given to illustrate the proposed approach in Section 4. Section 5 concludes the paper.

2. Preliminaries

2.1. Notation, definitions, basic result

The set of natural numbers, integers and real numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$, respectively. The set of nonnegative real numbers and nonnegative integers are denoted by $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$, respectively. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $|x|$, and for a measurable and locally essentially bounded input $u : \mathbb{Z} \to \mathbb{R}$, the symbol $\|u\|_{[t_0,t_1]}$ denotes its L_∞ norm,

$$\|u\|_{[t_0,t_1]} = \sup\{|u|, t \in [t_0, t_1]\}.$$

If $t_1 = \infty$ then we will simply write $\|u\|$. We denote $\mathcal{L}_\infty$ as the set of all inputs u with the property $\|u\| < \infty$. We denote the sequence of integers

$1, \ldots, N$ as $\overline{1, N}$. Inequalities must be understood *component-wise*, i.e., for $x_a = [x_{a,1}, ..., x_{a,n}]^\top \in \mathbb{R}^n$ and $x_b = [x_{b,1}, ..., x_{b,n}]^\top \in \mathbb{R}^n$, $x_a \leq x_b$ if and only if, for all $i \in \overline{1, N}$, $x_{a,i} \leq x_{b,i}$. For a square matrix $Q \in \mathbb{R}^{n\times n}$, let the matrix $Q^+ \in \mathbb{R}^{n\times n}$ denote $Q^+ = (\max\{q_{i,j}, 0\})_{i,j=1,1}^{n,n}$, where the notation $Q = (q_{i,j})_{i,j=1,1}^{n,n}$ is used. Let $Q^- \in \mathbb{R}^{n\times n}$ be defined by $Q^- = Q^+ - Q$ and the matrix of absolute values of all elements be defined by $|Q| = Q^+ + Q^-$, the superscripts $+$ and $-$ for other purposes are defined appropriately when they appear. A square matrix $Q \in \mathbb{R}^{n\times n}$ is said to be nonnegative if all its entries are nonnegative. I is the identity matrix of appropriate dimension. Any $n \times m$ (resp. $p \times 1$) matrix, whose entries are all 1 is denoted $E_{n\times m}$ (resp. E_p) and whose entries are all 0 is denoted $0_{n\times m}$ (resp. 0_p). The vector of eigenvalues of a matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $\lambda(A)$. A positive (res. negative) (semi) definte matrix $P \in \mathbb{R}^{n\times n}$ is denoted as $P \succ (\succcurlyeq)\ 0$ (resp. $P \prec (\preccurlyeq)\ 0$).

Consider $\underline{x}$, $\overline{x} \in \mathbb{R}^n$ such $\underline{x} \leq \overline{x}$ and define $\overline{X}^T = [\overline{x} \ \ \underline{x}]$ and $\underline{X}^T = [\underline{x} \ \ \overline{x}]$. For a non-square matrix B, the left pseudo-inverse of matrix B is $B^\oplus = (B^T B)^{-1} B^T$. Additionally, B^* is a matrix such that $B^* B = 0$.

Lemma 1. *[5] Consider a vector $x \in \mathbb{R}^n$ such that $\underline{x} \leq x \leq \overline{x}$ and a constant matrix $A \in \mathbb{R}^{n\times n}$, then*

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}, \tag{1}$$

with $A^+ = \max\{0, A\}$, $A^- = A^+ - A$. If A is satisfying the relation $\underline{A} \leq A \leq \overline{A}$, then

$$\underline{A}^+\underline{x}^+ - \overline{A}^+\underline{x}^- - \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^- \leq Ax \leq \overline{A}^+\overline{x}^+ - \underline{A}^+\overline{x}^- - \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^-.$$

Lemma 2. *[4, 8] Consider a positive scalar δ and a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, then*

$$2x^T y \leq \frac{1}{\delta} x^T P x + \delta y^T P^{-1} y,\ \ x, y \in \mathbb{R}^n. \tag{2}$$

Definition 1. *[35] A discrete-time system described by $x(k+1) = f(x(k))$ is nonnegative if for any integer k_0 and any initial condition $x(k_0) \geq 0$, the solution x satisfies $x(k) \geq 0$ for all integers $k \geq k_0$.*

Lemma 3. *[35] A system described by $x(k+1) = Ax(k) + u(k)$, with $x(k) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$, is nonnegative if and only if the matrix A is elementwise nonnegative, $u(k) \geq 0$ and $x(k_0) \geq 0$. In this case, the system is also called cooperative.*

The Lemma 3 is essential in the design of interval observers since the estimation errors should follow nonnegative dynamics. Without any lost of generality in the present paper, we choose $k_0 = 0$.

2.2. Average dwell time

Definition 2. *[25] For a switching signal σ and any $0 \leq k_l \leq k_s$, let $N_\sigma(k_l, k_s)$ denote the number of discontinuities of σ on the interval $[k_l, k_s)$. If there exist a scalar $\tau_a > 0$ and an integer $N_0 \geq 0$, such that*

$$N_\sigma(k_l, k_s) \leq N_0 + \frac{k_s - k_l}{\tau_a} \tag{3}$$

holds for all k_l and k_s, then the scalar $\tau_a > 0$ is called an average dwell time (ADT) and N_0 the chatter bound. In this paper, we assume that $N_0 = 0$ for simplicity as commonly used in the literature.

2.3. Input to state stability

Input-to-State Stability (ISS) is an approach to analyse the effect of external disturbance on the stability of systems. The following Lemma gives sufficient conditions on Input-to-State Stability for discrete time switched systems using multiple Lyapunov function.

Definition 3. *[23] A function φ is said to belong to the class $\mathcal{K}$ if $\varphi \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$, $\varphi(0) = 0$ and φ is strictly increasing. $\mathcal{K}_\infty$ is the subset of $\mathcal{K}$ functions that are unbounded. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is of class $\mathcal{KL}$, if $\beta(., t)$ is of class $\mathcal{K}$ in the first argument for each fixed $t \geq 0$ and $\beta(s, t)$ decreases to 0 as $t \longrightarrow +\infty$ for each fixed $s \geq 0$.*

Lemma 4. *[26] Consider the discrete-time switched system $x(k+1) = f_{\sigma(k)}(\xi(k), \eta(k))$, $\sigma(k) \in \overline{1,N}$. Suppose that there exists $\mathcal{C}^1$ functions $V_{\sigma(k)} : \mathbb{R}^n \longrightarrow \mathbb{R}_+$, class $\mathcal{K}_\infty$ functions α_1, α_2, γ and constants $0 < \alpha < 1$, $\mu \geq 1$ such that $\forall \xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^l$ we have*

$$\alpha_1(\|\xi\|) \leq V_{\sigma(k)}(\xi) \leq \alpha_2(\|\xi\|), \tag{4}$$

$$V_{\sigma(k)}(\xi(k+1)) - V_{\sigma(k)}(\xi(k)) \leq -\alpha V_{\sigma(k)}(\xi(k)) + \varrho(\|\eta\|), \tag{5}$$

and for each switching instant k_l, $l = 0, 1\ 2, 3, \ldots$,

$$V_{\sigma(k_l)}(\xi(k)) \leq \mu V_{\sigma(k_l - 1)}(\xi(k)). \tag{6}$$

Then the system $x(k+1) = f_{\sigma(k)}(\xi(k), \eta(k))$, $\sigma(k) \in \overline{1,N}$ is Input-to-State Stable for any switching signal satisfying the average dwell time

$$\tau_a \geq \tau_a^* = -\frac{\ln(\mu)}{\ln(1-\alpha)}. \tag{7}$$

3. Main results

Consider the following discrete-time linear switched system

$$\begin{cases} x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) + D_{\sigma(k)}d(k) + \omega(k), \\ y_m(k) = C_{\sigma(k)}x(k) + v(k), \ \ \sigma(k) \in \overline{1,N}, \ \ N \in \mathbb{N} \end{cases} \tag{8}$$

with $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input, $y_m \in \mathbb{R}^p$ is the output, $\omega \in \mathbb{R}^n$ and $v \in \mathbb{R}^p$ are respectively the disturbances and the measurement noises, $d \in \mathbb{R}^l$ is the unknown input. $\sigma(k) = \sigma_k, k = 0, 1, 2, 3, \ldots$ is a known piecewise constant function that takes its values in an index set $\overline{1,N}$, $N > 1$, where σ_k is the index of the active subsystem and N is the number of subsystems. A_{σ_k}, B_{σ_k} and C_{σ_k} and D_{σ_k} are time-invariant matrices with appropriate dimensions. Then for simplicity, (8) can be rewritten as

$$\begin{cases} x(k+1) = A_{\sigma_k}x(k) + B_{\sigma_k}u(k) + D_{\sigma_k}d(k) + \omega(k), \\ y_m(k) = C_{\sigma_k}x(k) + v(k), \ \ \sigma_k \in \overline{1,N}, \ \ N \in \mathbb{N} \end{cases} \tag{9}$$

The switched system (9) is affected by unknown disturbances on both the input and the output. In the case where such terms can not be measured, an unknown input estimator can be referred as a solution. In the presence of uncertainties which are unknown but bounded by known bounds, the use of classical observers is limited. However, interval observers can be considered as potential candidates to cope with such uncertainties and a joint estimation of the state and the unknown input may be performed in such a case.

Based on [7], the first step consists in employing a state transformation to decompose each mode of the system (9) into two subsystems where the first subsystem dynamics are completely decoupled from the unknown input. Therefore, such decoupled modes can be used to design an interval observer which allows one, with another transformation if necessary to relax the classical assumptions on the positivity of the estimation errors, to estimate the bounds of the state vector x. The second step consists in computing two bounds $\underline{d}$ and $\overline{d}$ for the unknown input vector d, satisfying

$$\underline{d}(k) \leq d(k) \leq \overline{d}(k),\ k \in \mathbb{Z}_+. \tag{10}$$

For the rest of the paper, some assumptions are introduced.

Assumption 1. *The switching signal $\sigma(k)$ is assumed to be known.*

Assumption 2. *The state disturbance and the noise measurement are assumed to be bounded such that*

$$-\overline{\omega} \leq \omega(k) \leq \overline{\omega},\ \forall k \geq 0, \tag{11}$$

$$-\overline{v} \leq v(k) \leq \overline{v},\ \forall k \geq 0, \tag{12}$$

with $\overline{\omega} \in \mathbb{R}^n$ and $\overline{v} \in \mathbb{R}^p$.

Assumption 3.

$$rank(C_{\sigma_k} D_{\sigma_k}) = rank(D_{\sigma_k}) = l, \forall \sigma_k \in \overline{1, N},\ l \leq p. \tag{13}$$

Remark 1. *Assumption 2 is realistic: it is frequently satisfied in practice. It can be relaxed by allowing the bounds to depend on time k but for the sake of simplicity, they are assumed to be constant.*
Assumption 3, also called a relative degree condition, is common in the unknown input observers literature. It establishes the extistence condition of an unknown input observer for the system (9).

3.1. Step 1: Unknown input decoupling

Based on Assumption 3 and [12], there exists a nonsingular state transformation

$$T_{\sigma_k} = \begin{bmatrix} D^*_{\sigma_k} \\ \\ (C_{\sigma_k} D_{\sigma_k})^{\oplus} C_{\sigma_k} \end{bmatrix} = \begin{bmatrix} D^*_{\sigma_k} \\ \\ \tilde{T}_{\sigma_k} \end{bmatrix}, \ T_{\sigma_k} \in \mathbb{R}^{n \times n}, \tag{14}$$

where $D^*_{\sigma_k}$ chosen such that $D^*_{\sigma_k} D_{\sigma_k} = 0$ and $(C_{\sigma_k} D_{\sigma_k})^{\oplus}$ is the left pseudo-inverse of $(C_{\sigma_k} D_{\sigma_k})$. The inverse matrix $T^{-1}_{\sigma_k}$ takes the following form

$$T^{-1}_{\sigma_k} = \begin{bmatrix} (I - D_{\sigma_k}(C_{\sigma_k} D_{\sigma_k})^{\oplus} C_{\sigma_k})(D^*_{\sigma_k})^{\oplus} & D_{\sigma_k} \end{bmatrix}. \tag{15}$$

Given the change of coordinates $z = T_{\sigma_k} x$, the system (9) becomes

$$\begin{cases} z(k+1) = \tilde{A}_{\sigma_k} z(k) + \tilde{B}_{\sigma_k} u(k) + \tilde{D}_{\sigma_k} d(k) + \tilde{w}_{\sigma_k}(k), \\ \\ y_m(k) = \tilde{C}_{\sigma_k} z(k) + v(k), \ \forall \sigma_k \in \overline{1, N}, \ N \in \mathbb{N}, \end{cases} \tag{16}$$

where

$$\tilde{A}_{\sigma_k} = T_{\sigma_k} A_{\sigma_k} T^{-1}_{\sigma_k} = \begin{bmatrix} \tilde{A}_{1\sigma_k} & \tilde{A}_{2\sigma_k} \\ \tilde{A}_{3\sigma_k} & \tilde{A}_{4\sigma_k} \end{bmatrix}, \ \tilde{C}_{\sigma_k} = C_{\sigma_k} T^{-1}_{\sigma_k},$$

$$\tilde{B}_{\sigma_k} = T_{\sigma_k} B_{\sigma_k} = \begin{bmatrix} \tilde{B}_{1\sigma_k} \\ \tilde{B}_{2\sigma_k} \end{bmatrix}, \ \tilde{D}_{\sigma_k} = T_{\sigma_k} D_{\sigma_k} = \begin{bmatrix} 0 \\ I_l \end{bmatrix},$$

$$\tilde{\omega}_{\sigma_k}(k) = T_{\sigma_k} \omega(k) = \begin{bmatrix} \tilde{\omega}_{1\sigma_k}(k), \\ \tilde{\omega}_{2\sigma_k} \end{bmatrix}, \ z(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix}, \ z_1 \in \mathbb{R}^{n-l}, \ z_2 \in \mathbb{R}^l.$$

Consequently, the system (16) is decomposed into unknown input-depending and unknown input-free subsystems respectively whose dynamics are described by

$$\begin{cases} z_1(k+1) = \tilde{A}_{1\sigma_k} z_1(k) + \tilde{A}_{2\sigma_k} z_2(k) + \tilde{B}_{1\sigma_k} u(k) + \tilde{\omega}_{1\sigma_k}(k), \\ z_2(k+1) = \tilde{A}_{3\sigma_k} z_1(k) + \tilde{A}_{4\sigma_k} z_2(k) + \tilde{B}_{2\sigma_k} u(k) + d(k) + \tilde{\omega}_{2\sigma_k}(k), \\ y_m(k) = \tilde{C}_{\sigma_k} z(k) + v(k). \end{cases} \tag{17}$$

Introducing the output transformation given by the following equation

$$\tilde{y}(k) = U_{\sigma_k} y_m(k), \tag{18}$$

with

$$U_{\sigma_k} = \begin{bmatrix} U_{1\sigma_k} \\ \\ U_{2\sigma_k} \end{bmatrix} = \begin{bmatrix} (C_{\sigma_k} D_{\sigma_k})^* \\ \\ (C_{\sigma_k} D_{\sigma_k})^{\oplus} \end{bmatrix}, \tag{19}$$

the decomposed system (17) becomes

$$\begin{cases} z_1(k+1) = \tilde{A}_{1\sigma_k} z_1(k) + \tilde{A}_{2\sigma_k} z_2(k) + \tilde{B}_{1\sigma_k} u(k) + \tilde{\omega}_{1\sigma_k}(k) \\ z_2(k+1) = \tilde{A}_{3\sigma_k} z_1(k) + \tilde{A}_{4\sigma_k} z_2(k) + \tilde{B}_{2\sigma_k} u(k) + d(k) + \tilde{\omega}_{2\sigma_k}(k) \\ \tilde{y}_1(k) = \check{C}_{\sigma_k} z_1(k) + U_{1\sigma_k} v(k) \\ \tilde{y}_2(k) = z_2(k) + U_{2\sigma_k} v(k) \end{cases}, \tag{20}$$

where the new form of the output is described by the following form

$$\tilde{y} = \begin{bmatrix} \tilde{y}_1 \\ \\ \tilde{y}_2 \end{bmatrix}, \ \check{C}_{\sigma_k} = (C_{\sigma_k} D_{\sigma_k})^* C_{\sigma_k} (D^*_{\sigma_k})^{\oplus}.$$

Because of the time-varying form of the transformations (14) and (19), the state $z(k)$ of the transformed system is governed by the following reset equation at the switching instants $k = k_l$

$$z(k_l) = T_{\sigma_{k_l}} x(k_l). \tag{21}$$

Using the relation of $\tilde{y}_2(k) = z_2(k) + U_{2\sigma_k} v(k)$ in (20), it follows that

$$z_2(k) = \tilde{y}_2(k) - U_{2\sigma_k} v(k). \tag{22}$$

The substitution of (22) into dynamics of z_1 in (20) yields

$$\begin{cases} z_1(k+1) &= \tilde{A}_{1\sigma_k} z_1(k) + \tilde{A}_{2\sigma_k} \tilde{y}_2(k) + \tilde{B}_{1\sigma_k} u(k) \\ & \quad + \tilde{\omega}_{1\sigma_k}(k) - \tilde{A}_{2\sigma_k} U_{2\sigma_k} v(k) \\ \\ \tilde{y}_1(k) &= \check{C}_{\sigma_k} z_1(k) + U_{1\sigma_k} v(k). \end{cases} \tag{23}$$

Assumption 4. *The pairs* $(\tilde{A}_{1\sigma_k},\ \check{C}_{\sigma_k})$ *are detectable for all* $\sigma_{\sigma_k} \in \overline{1, N}$.

Remark 2. *It is shown in [13] that the strong detectability of the matrix triplets* $(A_{\sigma_k}, C_{\sigma_k}, D_{\sigma_k})$ *is equivalent to the Assumption 4. Moreover, Assumption 4 is a necessary but not a sufficient condition for the existence of an interval observer for (23). An additional assumption related to the average dwell time condition will be presented later in Theorem 1.*

3.2. Step 2: Interval observer design for the unknown input-free subsystem

To design an interval observer, two properties have to be satisfied: (i) framer property which is the notion of providing intervals in which the state variables stay and (ii) stability property which cares the length of estimated intervals. For that reason, observer gains L_{σ_k} need to be chosen such that the matrices $\tilde{A}_{1\sigma_k} - L_{\sigma_k} \check{C}_{\sigma_k}$ are nonnegative and the estimation errors are stable, which is usually difficult. Naturally, one can think about finding a nonsingular transformation $\beta_1 = P z_1$ such that the matrices $P(\tilde{A}_{1\sigma_k} - L_{\sigma_k} \check{C}_{\sigma_k})P^{-1}$ are nonnegative. Subsequently, a framer can be constructed in these new coordinates. Nevertheless, the existence of a common transformation P for all $\sigma_k \in \overline{1, N}$ is not obvious, even impossible.

3.2.1. Framer design

In this subsection, a new methodology is proposed for the unknown input-free switched subsystem (23). It is based on the design, in the original coordinates of two conventional observers. The structure is inspired by the one proposed in [34]. The framer is given by the following equations:

$$
\begin{cases}
\hat{z}_1^+(k+1) & = (\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k})\hat{z}_1^+(k) + \tilde{B}_{1\sigma_k}u(k) + P_{\sigma_k}^{-1}|P_{\sigma_k}|\overline{\tilde{\omega}}_{1\sigma_k} \\
& +P_{\sigma_k}^{-1}\left[P_{\sigma_k}^+\left(\tilde{A}_{2\sigma_k}^+\overline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\underline{\tilde{y}}_2\right) - P_{\sigma_k}^-\left(\tilde{A}_{2\sigma_k}^+\underline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\overline{\tilde{y}}_2\right)\right] \\
& +P_{\sigma_k}^{-1}|P_{\sigma_k}||\tilde{A}_{2\sigma_k}U_{2\sigma_k}|\overline{v} + L_{\sigma_k}\tilde{y}_1 + P_{\sigma_k}^{-1}|P_{\sigma_k}||L_{\sigma_k}U_{1\sigma_k}|\overline{v}, \\
\\
\hat{z}_1^-(k+1) & = (\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k})\hat{z}_1^-(k) + \tilde{B}_{1\sigma_k}u(k) + P_{\sigma_k}^{-1}\left(-|P_{\sigma_k}|\right)\overline{\tilde{\omega}}_{1\sigma_k} \\
& +P_{\sigma_k}^{-1}\left[P_{\sigma_k}^+\left(\tilde{A}_{2\sigma_k}^+\underline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\overline{\tilde{y}}_2\right) - P_{\sigma_k}^-\left(\tilde{A}_{2\sigma_k}^+\overline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\underline{\tilde{y}}_2\right)\right] \\
& -P_{\sigma_k}^{-1}|P_{\sigma_k}||\tilde{A}_{2\sigma_k}U_{2\sigma_k}|\overline{v} + L_{\sigma_k}\tilde{y}_1 - P_{\sigma_k}^{-1}|P_{\sigma_k}||L_{\sigma_k}U_{1\sigma_k}|\overline{v},
\end{cases}
\tag{24}
$$

The proposed framer and its design conditions are given in the following theorem

Theorem 1. *Let Assumptions 1-4 be satisfied and* $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$. *Given the nonsingular transformation matrices* $P_{\sigma_k} \in \mathbb{R}^{(n-l)\times(n-l)}$ *such that* $P_{\sigma_k}(\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k})P_{\sigma_k}^{-1}$ *are nonnegative and consider the suitably selected initial conditions*

$$
\begin{cases}
\hat{z}_1^+(0) = P_{\sigma_k}^{-1}\left(P_{\sigma_k}^+\overline{z}_1(0) - P_{\sigma_k}^-\underline{z}_1(0)\right), \\
\hat{z}_1^-(0) = P_{\sigma_k}^{-1}\left(P_{\sigma_k}^+\underline{z}_1(0) - P_{\sigma_k}^-\overline{z}_1(0)\right),
\end{cases}
\tag{25}
$$

where

$$
\begin{cases}
\overline{z}(0) = T_{\sigma_0}^+\overline{x}(0) - T_{\sigma_0}^-\underline{x}(0), \\
\underline{z}(0) = T_{\sigma_0}^+\underline{x}(0) - T_{\sigma_0}^-\overline{x}(0),
\end{cases}
\tag{26}
$$

Then, the bounds of the substate vector z_1 *given by*

$$
\begin{cases}
\overline{z}_1(k) = (P_{\sigma_k}^{-1})^+P_{\sigma_k}\hat{z}_1^+(k) - (P_{\sigma_k}^{-1})^-P_{\sigma_k}\hat{z}_1^-(k), \\
\underline{z}_1(k) = (P_{\sigma_k}^{-1})^+P_{\sigma_k}\hat{z}_1^-(k) - (P_{\sigma_k}^{-1})^-P_{\sigma_k}\hat{z}_1^+(k),
\end{cases}
\tag{27}
$$

satisfy

$$
\underline{z}_1(k) \leq z_1(k) \leq \overline{z}_1(k), \ \forall\ k \geq 0. \tag{28}
$$

Proof. We have to prove that $\overline{z}_1(k) - z_1(k) \geq 0$ and $z_1(k) - \underline{z}_1(k) \geq 0,\ \forall\ k \geq 0$. Consider the upper and lower observation errors defined respectively by $\overline{e}_{z_1}(k) = P_{\sigma_k}\hat{z}_1^+(k) - P_{\sigma_k}z_1(k)$ and $\underline{e}_{z_1}(k) = P_{\sigma_k}z_1(k) - P_{\sigma_k}\hat{z}_1^-(k)$. Then, their dynamics can be expressed as

$$\begin{aligned}
\overline{e}_{z_1}(k+1) &= P_{\sigma_k}\hat{z}_1^+(k+1) - P_{\sigma_k}z_1(k+1) \\
&= P_{\sigma_k}(\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k})\hat{z}_1^+(k) - P_{\sigma_k}(\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k})z_1(k) \\
&+ \left[P_{\sigma_k}^+\left(\tilde{A}_{2\sigma_k}^+\overline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\underline{\tilde{y}}_2\right) - P_{\sigma_k}^-\left(\tilde{A}_{2\sigma_k}^+\underline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\overline{\tilde{y}}_2\right)\right] \\
&+ P_{\sigma_k}\tilde{A}_{2\sigma_k}U_{2\sigma_k}v + |P_{\sigma_k}|\overline{\tilde{\omega}}_{1\sigma_k} - P_{\sigma_k}\tilde{\omega}_{1\sigma_k} - P_{\sigma_k}\tilde{A}_{2\sigma_k}\tilde{y}_2(k) \\
&+ |P_{\sigma_k}||\tilde{A}_{2\sigma_k}U_{2\sigma_k}|\overline{v} + |P_{\sigma_k}||L_{\sigma_k}U_{1\sigma_k}|\overline{v} + P_{\sigma_k}L_{\sigma_k}U_{1\sigma_k}v \\
&= P_{\sigma_k}(\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k})\overline{e}_{z_1}(k) + \Upsilon_{\sigma_k}^+
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
\Upsilon_{\sigma_k}^+ &= |P_{\sigma_k}|\overline{\tilde{\omega}}_{1\sigma_k} - P_{\sigma_k}\tilde{\omega}_{1\sigma_k} + P_{\sigma_k}\tilde{A}_{2\sigma_k}U_{2\sigma_k}v + P_{\sigma_k}L_{\sigma_k}U_{1\sigma_k}v \\
&+ \left[P_{\sigma_k}^+\left(\tilde{A}_{2\sigma_k}^+\overline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\underline{\tilde{y}}_2\right) - P_{\sigma_k}^-\left(\tilde{A}_{2\sigma_k}^+\underline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\overline{\tilde{y}}_2\right)\right] \\
&- P_{\sigma_k}\tilde{A}_{2\sigma_k}\tilde{y}_2(k) + |P_{\sigma_k}||\tilde{A}_{2\sigma_k}U_{2\sigma_k}|\overline{v} + |P_{\sigma_k}||L_{\sigma_k}U_{1\sigma_k}|\overline{v}
\end{aligned}$$

The lower observation error is given by

$$\begin{aligned}
\underline{e}_{z_1}(k+1) &= P_{\sigma_k}z_1(k+1) - P_{\sigma_k}\hat{z}_1^-(k+1) \\
&= P_{\sigma_k}(\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k})\hat{z}_1(k) - P_{\sigma_k}(\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k})z_1^-(k) \\
&- \left[P_{\sigma_k}^+\left(\tilde{A}_{2\sigma_k}^+\underline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\overline{\tilde{y}}_2\right) - P_{\sigma_k}^-\left(\tilde{A}_{2\sigma_k}^+\overline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\underline{\tilde{y}}_2\right)\right] \\
&- P_{\sigma_k}\tilde{A}_{2\sigma_k}U_{2\sigma_k}v + |P_{\sigma_k}|\overline{\tilde{\omega}}_{1\sigma_k} + P_{\sigma_k}\tilde{\omega}_{1\sigma_k} + P_{\sigma_k}\tilde{A}_{2\sigma_k}\tilde{y}_2(k) \\
&+ |P_{\sigma_k}||\tilde{A}_{2\sigma_k}U_{2\sigma_k}|\overline{v} + |P_{\sigma_k}||L_{\sigma_k}U_{1\sigma_k}|\overline{v} - P_{\sigma_k}L_{\sigma_k}U_{1\sigma_k}v \\
&= P_{\sigma_k}(\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k})\underline{e}_{z_1}(k) + \Upsilon_{\sigma_k}^-
\end{aligned} \tag{30}$$

with

$$\begin{aligned}
\Upsilon_{\sigma_k}^- &= |P_{\sigma_k}|\overline{\tilde{\omega}}_{1\sigma_k} + P_{\sigma_k}\tilde{\omega}_{1\sigma_k} - P_{\sigma_k}\tilde{A}_{2\sigma_k}U_{2\sigma_k}v - P_{\sigma_k}L_{\sigma_k}U_{1\sigma_k}v \\
&- \left[P_{\sigma_k}^+\left(\tilde{A}_{2\sigma_k}^+\underline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\overline{\tilde{y}}_2\right) - P_{\sigma_k}^-\left(\tilde{A}_{2\sigma_k}^+\overline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\underline{\tilde{y}}_2\right)\right] \\
&+ P_{\sigma_k}\tilde{A}_{2\sigma_k}\tilde{y}_2(k) + |P_{\sigma_k}||\tilde{A}_{2\sigma_k}U_{2\sigma_k}|\overline{v} + |P_{\sigma_k}||L_{\sigma_k}U_{1\sigma_k}|\overline{v}
\end{aligned}$$

Taking in mind Lemma 1, the following inclusions hold

$$\Gamma^- \leq P_{\sigma_k}\tilde{A}_{2\sigma_k}\tilde{y}_2(k) \leq \Gamma^+ \tag{31}$$

with

$$\Gamma^- = P_{\sigma_k}^+\left(\tilde{A}_{2\sigma_k}^+\underline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\overline{\tilde{y}}_2\right) - P_{\sigma_k}^-\left(\tilde{A}_{2\sigma_k}^+\overline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\underline{\tilde{y}}_2\right)$$

and

$$\Gamma^+ = P_{\sigma_k}^+\left(\tilde{A}_{2\sigma_k}^+\overline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\underline{\tilde{y}}_2\right) - P_{\sigma_k}^-\left(\tilde{A}_{2\sigma_k}^+\underline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\overline{\tilde{y}}_2\right)$$

$$-|P_{\sigma_k}||\tilde{A}_{2\sigma_k}U_{2\sigma_k}|\overline{v} \leq P_{\sigma_k}\tilde{A}_{2\sigma_k}U_{2\sigma_k}v \leq |P_{\sigma_k}||\tilde{A}_{2\sigma_k}U_{2\sigma_k}|\overline{v} \tag{32}$$

$$-|P_{\sigma_k}|\overline{\tilde{\omega}}_{1\sigma_k} \leq P_{\sigma_k}\tilde{\omega}_{1\sigma_k} \leq |P_{\sigma_k}|\overline{\tilde{\omega}}_{1\sigma_k} \tag{33}$$

Based on (31),(32), (33), $\Upsilon^+_{\sigma_k}$, $\Upsilon^-_{\sigma_k}$ are nonnegative by construction, $\overline{e}_{z_1}(0) \geq 0$, $\underline{e}_{z_1}(0) \geq 0$ and $P_{\sigma_k}(\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k})P^{-1}_{\sigma_k}$ are nonnegative. Then, based on Lemma 3, the upper and lower errors $\overline{e}_{z_1}(k)$ and $\underline{e}_{z_1}(k)$ are nonnegative for all $k \geq 0$ such that

$$P_{\sigma_k}\hat{z}^-_1(k) \leq P_{\sigma_k}z_1(k) \leq P_{\sigma_k}\hat{z}^+_1(k) \tag{34}$$

Therefore, the upper and lower bounds of the substate z_1 given by (27) verify

$$\underline{z}_1(k) \leq z_1(k) \leq \overline{z}_1(k), \ \forall \ k \geq 0$$

□

3.2.2. Stability conditions

In this subsection, the stability of the interval observer (24)-(27) is investigated in the ISS framework.

Theorem 2. *Assume that the conditions of Theorem 1 are satisfied. If there exist positive scalars* $\alpha_2 > \alpha_1 > 0$, $\gamma > 0$, $0 < \alpha < 1$ *and* $0 \leq \beta \leq 1$, *matrices* W_{σ_l}, S_{σ_l} *and diagonal positive definite matrices* M_{σ_k} *such that for* $\sigma_{k,l} \in \overline{1,N}$ *with* $\sigma_k \neq \sigma_l$,

$$\begin{bmatrix} -(1-\alpha)M_{\sigma_k} & 0 & \tilde{A}^T_{1\sigma_k}M_{\sigma_k} - \check{C}^T_{\sigma_k}S_{\sigma_k} \\ 0 & -\gamma^2 I_n & M_{\sigma_k} \\ M_{\sigma_k}\tilde{A}_{1\sigma_k} - S_{\sigma_k}\check{C}_{\sigma_k} & M_{\sigma_k} & -M_{\sigma_k} \end{bmatrix} \preceq 0, \ \forall \sigma_k \in \overline{1,N} \tag{35}$$

$$\alpha_1 I_n \leq M_{\sigma_k} \leq \alpha_2 I_n \tag{36}$$

$$\begin{bmatrix} W_{\sigma_l} & M_{\sigma_k} \\ \\ M_{\sigma_k} & M_{\sigma_k} \end{bmatrix} \succeq 0 \tag{37}$$

then, the lower and upper observer errors are ISS and the framer (24)-(27) *is an interval observer. In addition, the gains* L_{σ_k}*, given by* $L_{\sigma_k} = M_{\sigma_k}^{-1} S_{\sigma_k}$ *can be computed by minimising the linear problem*

$$\begin{aligned} &\underset{M_{\sigma_k}, S_{\sigma_k}, W_{\sigma_l}}{\textit{minimize}} \quad \beta\mu + (1-\beta)\gamma,\ \sigma_{k,l} \in \overline{1, N} \\ &\textit{subject to} \qquad (35), (36), (37). \end{aligned} \tag{38}$$

Proof. Consider a Multiple Quadratic Lyapunov (MQLF) function for the estimation error $e^+(k) = \hat{z}_1^+(k) - z_1(k)$ defined as

$$V_{\sigma_k}(e^+) = {e^+}^T M_{\sigma_k} e^+, \tag{39}$$

where M_{σ_k} are diagonal positive definite matrices. As shown in the previous section, the dynamics $e^+(k)$ are described by

$$e^+(k+1) \quad = (\tilde{A}_{1\sigma_k} - L_{\sigma_k} \check{C}_{\sigma_k}) e^+(k) + P_{\sigma_k}^{-1} \Upsilon_{\sigma_k}^+ \tag{40}$$

For the sequel, let define $\Phi_{\sigma_k} = (\tilde{A}_{1\sigma_k} - L_{\sigma_k} \check{C}_{\sigma_k})$. Therfore the increment of the Lyapunov function (39) is given by

$$\begin{aligned} \Delta V_{\sigma_k}(e^+) \quad &= V_{\sigma_k}(e^+(k+1)) - V_{\sigma_k}(e^+(k)) \\ &= {e^+}^T(k+1) M_{\sigma_k} e^+(k+1) - {e^+}^T(k) M_{\sigma_k} e^+(k) \\ &= {e^+}^T(k) \left[\Phi_{\sigma_k}^T M_{\sigma_k} \Phi_{\sigma_k} - M_{\sigma_k}\right] e^+(k) \\ &+ {e^+}^T(k) \Phi_{\sigma_k}^T M_{\sigma_k} \Xi_{\sigma_k}^+ + {\Xi^+}^T M_{\sigma_k} \Phi_{\sigma_k} e^+(k) + {\Xi_{\sigma_k}^+}^T M_{\sigma_k} \Xi_{\sigma_k}^+ \end{aligned} \tag{41}$$

with $\Xi_{\sigma_k}^+ = P_{\sigma_k}^{-1} \Upsilon_{\sigma_k}^+$.

By adding and subtracting the terms $\alpha {e^+}^T(k) M_{\sigma_k} e^+(k) - \gamma^2 {\Xi_{\sigma_k}^+}^T \Xi_{\sigma_k}^+$ to (41), we obatain

$$\begin{aligned} \Delta V_{\sigma_k}(e^+(k)) \quad &= {e^+}^T(k) \left[\Phi_{\sigma_k}^T M_{\sigma_k} \Phi_{\sigma_k} - (1-\alpha) M_{\sigma_k}\right] e^+(k) \\ &+ {e^+}^T(k) \Phi_{\sigma_k}^T M_{\sigma_k} \Xi_{\sigma_k}^+ + {\Xi^+}^T M_{\sigma_k} \Phi_{\sigma_k} e^+(k) \\ &+ {\Xi_{\sigma_k}^+}^T M_{\sigma_k} \Xi_{\sigma_k}^+ - \gamma^2 {\Xi_{\sigma_k}^+}^T \Xi_{\sigma_k}^+ - \gamma^2 {\Xi_{\sigma_k}^+}^T \Xi_{\sigma_k}^+ \end{aligned} \tag{42}$$

Then (42) can be rewritten as

$$\begin{aligned}\Delta V(e^+(k)) &= \left[e^{+^T}(k)\ \Xi^{+^T}_{\sigma_k}\right]\Lambda_{\sigma_k}\left[e^+(k)\ \Xi^+_{\sigma_k}\right] \\ &\quad -\alpha e^{+^T}(k)M_{\sigma_k}e^+(k)-\gamma^2\Xi^{+^T}_{\sigma_k}\Xi^+_{\sigma_k}\end{aligned} \tag{43}$$

where

$$\Lambda_{\sigma_k}=\begin{bmatrix}\Phi^T_{\sigma_k}M_{\sigma_k}\Phi_{\sigma_k}-(1-\alpha)M_{\sigma_k} & \Phi^T_{\sigma_k}M_{\sigma_k}\\ \\ M_{\sigma_k}\Phi_{\sigma_k} & M_{\sigma_k}-\gamma^2 I_n\end{bmatrix},\ \forall\sigma_k\in\overline{1,N} \tag{44}$$

thus, (44) can be rewritten as follows

$$\Lambda_{\sigma_k}=\begin{bmatrix}-(1-\alpha)M_{\sigma_k} & 0\\ \\ 0 & -\gamma^2 I_n\end{bmatrix}+\begin{bmatrix}\Phi^T_{\sigma_k}M_{\sigma_k}\\ \\ M_{\sigma_k}\end{bmatrix}M^{-1}_{\sigma_k}\begin{bmatrix}M_{\sigma_k}\Phi_{\sigma_k} & M_{\sigma_k}\end{bmatrix} \tag{45}$$

Using the Schur complement, we obtain

$$\Lambda_{\sigma_k}=\begin{bmatrix}-(1-\alpha)M_{\sigma_k} & 0 & \Phi^T_{\sigma_k}M_{\sigma_k}\\ \\ 0 & -\gamma^2 I_n & M_{\sigma_k}\\ \\ M_{\sigma_k}\Phi_{\sigma_k} & M_{\sigma_k} & -M_{\sigma_k}\end{bmatrix}\preceq 0,\ \forall\sigma_k\in\overline{1,N} \tag{46}$$

Based on Lemma 4 and using (43), we arrive at

$$\Delta V(e^+(k))\quad < -\alpha e^{+^T}(k)M_{\sigma_k}e^+(k)+\gamma^2\parallel\Xi^+_{\sigma_k}\parallel_2^2 \tag{47}$$

Let the inequality (47) hold for $k\in[k_0,k)$, which implies that

$$V_{\sigma_k}(e^+(k))\quad <(1-\alpha)^{(k-k_0)}V_{\sigma_k}(e^+(k))+\sum_{m=0}^{k-k_0-1}(1-\alpha)^m\gamma^2\parallel\Xi^+_{\sigma_k}\parallel_2^2 \tag{48}$$

then the following inequality is deduced

$$\alpha_1(\parallel e^+(k)\parallel)\leq V_{\sigma_k}(e^+(k)). \tag{49}$$

Thus,

$$\parallel e^+(k)\parallel_2\leq\frac{1}{\alpha_1}\left((1-\alpha)^{(k-k_0)}V_{\sigma_k}(e^+(k))+\sum_{m=0}^{k-k_0-1}(1-\alpha)^m\gamma^2\parallel\Xi^+_{\sigma_k}\parallel_2^2\right) \tag{50}$$

Let Assumption 2 hold, then $\Xi^{+}_{\sigma_k}$ is bounded when $k \rightarrow \infty$, then $\|\Xi^{+}_{\sigma_k}\|_\infty \leq \Xi^{+}$. One can deduce that

$$\lim_{k \longrightarrow \infty} \parallel e^{+}(k) \parallel_2 < \frac{\gamma^2}{\alpha_1 \alpha} \Xi^{+2} \tag{51}$$

The expression (51) shows that the interval error width is bounded by $\dfrac{\gamma^2}{\alpha_1 \alpha} \Xi^{+2}$, which depends on γ for given α_1 and α.

Furthermore, the stability at the switching instants is guranteed based on (6) which yields

$$\mu M_{\sigma_l} - M_{\sigma_k} \succeq 0 \tag{52}$$

By appying the Schur complement, we get

$$\begin{bmatrix} \mu M_{\sigma_l} & I_n \\ I_n & M^{-1}_{\sigma_k} \end{bmatrix} \succeq 0 \tag{53}$$

Let us multiply the both sides by $\begin{bmatrix} I_n & 0_n \\ 0_n & M_{\sigma_k} \end{bmatrix}$, we have the following inequality

$$\begin{bmatrix} W_{\sigma_l} & M_{\sigma_k} \\ M_{\sigma_k} & M_{\sigma_k} \end{bmatrix} \succeq 0 \tag{54}$$

with $W_{\sigma_l} = \mu M_{\sigma_l}$.

By making a recursion for the inequality (5) over the interval $[k_l, k)$, one can write:

$$V_{\sigma_i}(e^{+}(k)) \leq (1-\alpha)^{k-k_l} V_{\sigma_i}(e^{+}(k_l)), \forall\ i \in \overline{1, N} \tag{55}$$

In addition, based on (6), we obtain at the switching time k_l,

$$V_{\sigma_{k_l}}(e^{+}(k)) \leq \mu V_{\sigma_{k_{l}-1}}(e^{+}(k)) \tag{56}$$

Let us define $\varsigma = N_{\sigma_k}(0, K)$, at instant K, by using (55) and (56) we can write

$$\begin{aligned} V_{\sigma_K}(e^+(K)) &\leq (1-\alpha)^{(K-\varsigma)} V_{\sigma_{k_\varsigma}}(e^+(k_\varsigma)) \\ &\leq \mu (1-\alpha)^{(K-\varsigma)} V_{\sigma_{k_\varsigma - 1}}(e^+(k_\varsigma - 1)) \\ &\vdots \\ &\leq \mu^\varsigma (1-\alpha)^K V_{\sigma_0}(e^+(0)) \\ \\ &= \left((1-\alpha)\, \mu^{\frac{1}{\tau_a}}\right)^K V_{\sigma_0}(e^+(0)) \end{aligned} \tag{57}$$

Therefore, if the average dwell time satisfies (7) we obtain

$$(1-\alpha)\, \mu^{\frac{1}{\tau_a}} \quad \leq (1-\alpha)\, \mu^{-\frac{\ln(1-\alpha)}{\ln(\mu)}} \quad \leq \frac{1-\alpha}{1-\alpha} = 1 \tag{58}$$

Let equations (4), (5), and (55) hold, then

$$\begin{aligned} V_{\sigma_k}(e^+(k)) &\leq (1-\alpha)^{k-k_l} V_{\sigma_k}(e^+(k)) \\ &\leq (1-\alpha)^{k-k_l} \frac{V_{\sigma_k}(e^+(k))}{V_{\sigma_l}(e^+(k))} V_{\sigma_l}(e^+(k)) \\ &\leq \frac{\alpha_2}{\alpha_1} (1-\alpha)^{k-k_l} V_{\sigma_l}(e^+(k)) \end{aligned} \tag{59}$$

At switching time, $k = k_l$,

$$V_{\sigma_k}(e^+(k)) \quad \leq \frac{\alpha_2}{\alpha_1} V_{\sigma_l}(e^+(k)) \tag{60}$$

with $\mu = \dfrac{\alpha_2}{\alpha_1}$.

Then, the ISS conditions presented in Lemma 4 are verified for e^+. Note that $\overline{e}_{z_1} = P_{\sigma_k} \hat{z}_1^+ - P_{\sigma_k} z_1$ and since P_{σ_k} is bounded, one can deduce that $\overline{e}_{z_1}$ is also bounded. The same arguments show the the stability of the estimation error e^- and thus $\underline{e}_{z_1}$ is bounded, therefore, (24)-(27) represent an interval observer for (23).

An optimum average dwell time is fulfilled by defining an objective function added to LMI conditions. As presented in [3], this optimum is ensured by minimizing μ in the following objective function

$$\beta\mu + (1-\beta)\gamma \tag{61}$$

with $\beta \ \in [0,\ 1]$. □

3.3. Interval state estimation in the original coordinates

Based on the estimation of the state in the coordinates z_1, the bounds $\underline{x}$ and $\overline{x}$ are deduced in the following theorem.

Theorem 3. *Let the assumptions of Theorem 1 and Theorem 2 hold, then*

$$\underline{x}(k) \leq x(k) \leq \overline{x}(k), \ \forall \ k \geq 0 \tag{62}$$

where

$$\begin{cases} \overline{x}_1 & = T^+_{1\sigma_k}\overline{z}_1 - T^-_{1\sigma_k}\underline{z}_1 + T^+_{2\sigma_k}\overline{\tilde{y}}_2 - T^-_{2\sigma_k}\underline{\tilde{y}}_2 \\ & +(-T_{2\sigma_k}U_{2\sigma_k})^+\overline{v} + (-T_{2\sigma_k}U_{2\sigma_k})^-\overline{v} \\ \\ \underline{x}_1 & = T^+_{1\sigma_k}\underline{z}_1 - T^1_{1\sigma_k}\overline{z}_1 + T^+_{2\sigma_k}\underline{\tilde{y}}_2 - T^-_{2\sigma_k}\overline{\tilde{y}}_2 \\ & -(-T_{2\sigma_k}U_{2\sigma_k})^+\overline{v} - (-T_{2\sigma_k}U_{2\sigma_k})^-\overline{v} \\ \\ \overline{x}_2 & = T^+_{3\sigma_k}\overline{z}_1 - T^1_{3\sigma_k}\underline{z}_1 + T^+_{4\sigma_k}\overline{\tilde{y}}_2 - T^-_{4\sigma_k}\underline{\tilde{y}}_2 \\ & +(-T_{4\sigma_k}U_{2\sigma_k})^+\overline{v} + (-T_{4\sigma_k}U_{2\sigma_k})^-\overline{v} \\ \\ \underline{x}_2 & = T^+_{3\sigma_k}\underline{z}_1 - T^1_{3\sigma_k}\overline{z}_1 + T^+_{4\sigma_k}\underline{\tilde{y}}_2 - T^-_{4\sigma_k}\overline{\tilde{y}}_2 \\ & -(-T_{4\sigma_k}U_{2\sigma_k})^+\overline{v} - (-T_{4\sigma_k}U_{2\sigma_k})^-\overline{v} \end{cases} \tag{63}$$

$$\underline{x}(k) = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}, \ \overline{x}(k) = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix}.$$

Proof. Recaling that $x = T_{\sigma_k}^{-1} z$, with $T_{\sigma_k}^{-1} = \begin{bmatrix} T_{1\sigma_k} & T_{2\sigma_k} \\ T_{3\sigma_k} & T_{4\sigma_k} \end{bmatrix}$, then

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} T_{1\sigma_k} & T_{2\sigma_k} \\ T_{3\sigma_k} & T_{4\sigma_k} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ &= \begin{bmatrix} T_{1\sigma_k} & T_{2\sigma_k} \\ T_{3\sigma_k} & T_{4\sigma_k} \end{bmatrix} \begin{bmatrix} z_1 \\ \tilde{y_2} - U_{2\sigma_k} v \end{bmatrix} \\ &= \begin{bmatrix} T_{1\sigma_k} z_1 + T_{2\sigma_k} \tilde{y_2} - T_{2\sigma_k} U_{2\sigma_k} v \\ T_{3\sigma_k} z_1 + T_{4\sigma_k} \tilde{y_2} - T_{4\sigma_k} U_{2\sigma_k} v \end{bmatrix} \end{aligned}$$

Consider the following observation errors

$$\begin{cases} \overline{e}_{x_1} &= \overline{x}_1 - x_1 \\ \underline{e}_{x_1} &= x_1 - \underline{x}_1 \\ \overline{e}_{x_2} &= \overline{x}_2 - x_2 \\ \underline{e}_{x_2} &= x_2 - \underline{x}_2 \end{cases} \tag{64}$$

where

$$\begin{aligned} \overline{e}_{x_1} &= T_{1\sigma_k}^{+} \overline{E}_{z_1} + T_{1\sigma_k}^{-} \underline{E}_{z_1} + |T_{2\sigma_k} U_{1\sigma_k}| \overline{v} \\ &+ (-T_{2\sigma_k} U_{1\sigma_k})^{+} (\overline{v} - v) + (-T_{2\sigma_k} U_{1\sigma_k})^{-} (\overline{v} + v), \end{aligned}$$

$$\begin{aligned} \underline{e}_{x_1} &= T_{1\sigma_k}^{+} \underline{E}_{z_1} + T_{1\sigma_k}^{-} \overline{E}_{z_1} + |T_{2\sigma_k} U_{1\sigma_k}| \overline{v} \\ &+ (-T_{2\sigma_k} U_{1\sigma_k})^{+} (\overline{v} + v) + (-T_{2\sigma_k} U_{1\sigma_k})^{-} (\overline{v} - v), \end{aligned}$$

with

$$\overline{E}_{z_1} = \overline{z}_1 - z_1, \ \underline{E}_{z_1} = z_1 - \underline{z}_1.$$

Since, the observation errors $\overline{e}_{x_1}$, $\underline{e}_{x_1}$, $\overline{e}_{x_2}$, $\underline{e}_{x_2}$ are nonnegative, it yields

$$\underline{x}(k) \leq x(k) \leq \overline{x}(k)$$

To prove the boundedness of $\overline{x}_i$ and $\underline{x}_i$ with $i \in 1, 2$, let us consider the errors $e_{x_1} = \overline{x}_1 - \underline{x}_1$ and $e_{x_2} = \overline{x}_2 - \underline{x}_2$

$$\begin{aligned} e_{x_1} &= \overline{x}_1 - \underline{x}_1 \\ &= |T_{1\sigma_k}|\overline{E}_{z_1} + |T_{1\sigma_k}|\underline{E}_{z_1} + 4|T_{2\sigma_k}U_{1\sigma_k}|\overline{v}, \end{aligned} \tag{65}$$

and

$$\begin{aligned} e_{x_2} &= \overline{x}_2 - \underline{x}_2 \\ &= |T_{3\sigma_k}|\overline{E}_{z_1} + |T_{3\sigma_k}|\underline{E}_{z_1} + 4|T_{4\sigma_k}U_{1\sigma_k}|\overline{v}, \end{aligned} \tag{66}$$

Furthermore

$$\begin{aligned} \overline{E}_{z_1}(k) &= (P_{\sigma_k}^{-1})^{+}P_{\sigma_k}\hat{z}_1^{+}(k) - (P_{\sigma_k}^{-1})^{-}P_{\sigma_k}\hat{z}_1^{-}(k) - z_1(k) \\ &= (P_{\sigma_k}^{-1})^{+}P_{\sigma_k}e^{+}(k) + (P_{\sigma_k}^{-1})^{-}P_{\sigma_k}e^{-}(k), \end{aligned} \tag{67}$$

and

$$\begin{aligned} \underline{E}_{z_1}(k) &= z_1(k) - (P_{\sigma_k}^{-1})^{+}P_{\sigma_k}\hat{z}_1^{-}(k) + (P_{\sigma_k}^{-1})^{-}P_{\sigma_k}\hat{z}_1^{+}(k) \\ &= (P_{\sigma_k}^{-1})^{+}P_{\sigma_k}e^{-}(k) + (P_{\sigma_k}^{-1})^{-}P_{\sigma_k}e^{+}(k). \end{aligned} \tag{68}$$

Since, P_{σ_k} and $P_{\sigma_k}^{-1}$ are bounded for all $\sigma_k \in \overline{1, N}$, if (51) holds, one can deduce that $\overline{E}_{z_1}$ and $\underline{E}_{z_1}$ are bounded. Taking in mind the construction of e_{x_1} and e_{x_2}, then $\overline{x}_i$ and $\underline{x}_i$ with $i \in 1, 2$ are bounded. □

3.4. Unknown input estimation

The upper and the lower bounds of the unknown input d are given in the sequel. The dynamics of z_2 are given by

$$z_2(k+1) = U_{2\sigma_k}y_m(k+1) - U_{2\sigma_k}v(k+1) \tag{69}$$

Based on equation (17), the expression of the unknown input vector at time k is given by

$$\begin{aligned} d(k) &= z_2(k+1) - \tilde{A}_{3\sigma_k}z_1(k) - \tilde{A}_{4\sigma_k}z_2(k) - \tilde{B}_{2\sigma_k}u(k) - \tilde{\omega}_{2\sigma_k}(k) \\ &= U_{2\sigma_k}\left[y_m(k+1) - v(k+1)\right] - \tilde{A}_{3\sigma_k}z_1(k) - A_{4\sigma_k}U_{2\sigma_k}y_m \\ &+ A_{4\sigma_k}U_{2\sigma_k}v(k) - \tilde{B}_{2\sigma_k}u(k) - \tilde{\omega}_{2\sigma_k}(k) \end{aligned} \tag{70}$$

The upper and lower bounds of d given by (70) are expressed as

$$\begin{cases} \overline{d}(k) & = \left[U^{+}_{2\sigma_k}\overline{\chi}(k+1) - U^{-}_{2\sigma_k}\underline{\chi}(k+1)\right] - \tilde{B}_{2\sigma_k}u(k) \\ & + \left[(-\tilde{A}_{3\sigma_k})^{+}\overline{z}_1(k) - (-\tilde{A}_{3\sigma_k})^{-}\underline{z}_1(k)\right] + \overline{\tilde{\omega}}_{2\sigma_k} \\ & + \left[(-A_{4\sigma_k}U_{2\sigma_k})^{+}\overline{y}_m(k) - (-A_{4\sigma_k}U_{2\sigma_k})^{-}\underline{y}_m(k)\right] + |A_{4\sigma_k}U_{2\sigma_k}|\overline{v}, \\ \underline{d}(k) & = \left[U^{+}_{2\sigma_k}\underline{\chi}(k+1) - U^{-}_{2\sigma_k}\overline{\chi}(k+1)\right] - \tilde{B}_{2\sigma_k}u(k) \\ & + \left[(-\tilde{A}_{3\sigma_k})\underline{z}_1(k) - (-\tilde{A}_{3\sigma_k})^{-}\overline{z}_1(k)\right] - \overline{\tilde{\omega}}_{2\sigma_k} \\ & + \left[(-A_{4\sigma_k}U_{2\sigma_k})^{+}\underline{y}_m(k) - (-A_{4\sigma_k}U_{2\sigma_k})^{-}\overline{y}_m(k)\right] - |A_{4\sigma_k}U_{2\sigma_k}|\overline{v}, \end{cases} \tag{71}$$

with $\chi(k) = y_m(k) - v(k)$. Where $\overline{\chi}(k)$ and $\underline{\chi}(k)$ are respectively upper and lower bound of $\chi(k)$

$$\begin{cases} \overline{\chi}(k) & = y_m(k) + \overline{v} \\ \underline{\chi}(k) & = y_m(k) - \overline{v} \end{cases}. \tag{72}$$

Theorem 4. *Assume that the assumptions of Theorem 2 are satisfied, then, (73) is an interval estimation for the unknown signal d, such that*

$$\underline{d}(k) \leq d(k) \leq \overline{d}(k). \tag{73}$$

Proof. Let us define the upper and lower bound of the observation errors of the unknown input d as

$$\begin{cases} \overline{e}_d(k) & = \overline{d}(k) - d(k) \\ \underline{e}_d(k) & = d(k) - \underline{d}(k) \end{cases} \tag{74}$$

Using (71) and (74), we obtain

$$\begin{cases} \overline{e}_d(k) & = U^{+}_{2\sigma_k}\left[\overline{\chi}(k+1) - \chi(k+1)\right] + U^{-}_{2\sigma_k}\left[\chi(k+1) - \underline{\chi}(k+1)\right] \\ & +(-\tilde{A}_{3\sigma_k})^{+}\left[\overline{z}_1(k) - z_1(k)\right] + (-\tilde{A}_{3\sigma_k})^{-}\left[z_1(k) - \underline{z}_1(k)\right] \\ & +(-A_{4\sigma_k}U_{2\sigma_k})^{+}\left[\overline{y}_m(k) - y_m(k)\right] + (-A_{4\sigma_k}U_{2\sigma_k})^{-}\left[y_m(k) - \underline{y}_m(k)\right] \\ & +(A_{4\sigma_k}U_{2\sigma_k})^{+}\left[\overline{v} - v\right] + (A_{4\sigma_k}U_{2\sigma_k})^{-}\left[\overline{v} + v\right] + \left[\overline{\tilde{\omega}}_{2\sigma_k} + \tilde{\omega}_{2\sigma_k}\right] \\ \\ \underline{e}_d(k) & = U^{+}_{2\sigma_k}\left[\chi(k+1) - \underline{\chi}(k+1)\right] + U^{-}_{2\sigma_k}\left[\overline{\chi}(k+1) - \chi(k+1)\right] \\ & +(-\tilde{A}_{3\sigma_k})^{+}\left[z_1(k) - \underline{z}_1(k)\right] + (-\tilde{A}_{3\sigma_k})^{-}\left[\overline{z}_1(k) - z_1(k)\right] \\ & +(-A_{4\sigma_k}U_{2\sigma_k})^{+}\left[y_m(k) - \underline{y}_m(k)\right] + (-A_{4\sigma_k}U_{2\sigma_k})^{-}\left[\overline{y}_m(k) - y_m(k)\right] \\ & +(A_{4\sigma_k}U_{2\sigma_k})^{+}\left[\overline{v} - v\right] + (A_{4\sigma_k}U_{2\sigma_k})^{-}\left[\overline{v} + v\right] + \left[\overline{\tilde{\omega}}_{2\sigma_k} - \tilde{\omega}_{2\sigma_k}\right] \end{cases} \tag{75}$$

By analysing the construction of the bounds of the observation errors of the unknown input d given by (75), and taking in mind Assumptions 1-2, the upper and the lower observation errors of the unknown input $\overline{e}_d$ and $\underline{e}_d$ are nonnegative.

To prove their boundedness, consider $e_d = \overline{d} - \underline{d}$. Hence,

$$\begin{aligned} e_d &= U^+_{2\sigma_k}\left[\overline{\chi}(k+1) - \underline{\chi}(k+1)\right] + U^-_{2\sigma_k}\left[\overline{\chi}(k+1) - \underline{\chi}(k+1)\right] \\ &+(-\tilde{A}_{3\sigma_k})^+\left[\overline{z}_1(k) - \underline{z}_1(k)\right] + (-\tilde{A}_{3\sigma_k})^-\left[\overline{z}_1(k) - \underline{z}_1(k)\right] \\ &+(-A_{4\sigma_k}U_{2\sigma_k})^+\left[\overline{y}_m(k) - \underline{y}_m(k)\right] + (-A_{4\sigma_k}U_{2\sigma_k})^-\left[\overline{y}_m(k) - \underline{y}_m(k)\right] \\ &+2|A_{4\sigma_k}U_{2\sigma_k})|\overline{v} + 2\overline{\tilde{\omega}}_{2\sigma_k} \end{aligned} \tag{76}$$

Using results of Theorem 1 and Theorem 3 as well as taking in mind the construction of e_d, the boundedness of d is verified. □

4. Numerical simulations

Given the system (9) with three modes ($N = 3$) where:

$$A_1 = \begin{bmatrix} 0.55 & 0.5 & 0.7 \\ 0 & 0.8 & 0.5 \\ 0 & 0 & 0.4 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0.5 \\ 0.7 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, D_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.2 & -0.1 & 0.1 \\ 0 & 0.4 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix}, B_2 = \begin{bmatrix} 0.4 \\ 0.3 \\ 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1.01 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, D_2 = \begin{bmatrix} 1 \\ 0 \\ 4.73 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0.09 & 0.09 & 0.09 \\ 0.09 & 0.18 & 0.09 \\ 0.09 & 0.09 & 0.27 \end{bmatrix}, B_3 = \begin{bmatrix} 0.1 \\ 0.0 \\ 0.1 \end{bmatrix}, C_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, D_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

$w(k)$ and $v(k)$ are respectively the disturbances and the measurement noises which are uniformly bounded such that $-\overline{w} \leq w(k) \leq \overline{w}$ with $\overline{w} = \begin{bmatrix} 0.06 & 0.06 & 0.06 \end{bmatrix}$, and $-\overline{v} \leq v(k) \leq \overline{v}$ with $\overline{v} = \begin{bmatrix} 0.06 & 0.06 \end{bmatrix}$. The unknown input is given as

$d(k) = 0.5\sin(0.5k)$. As above-mentioned, the synthesis of the unknown input interval estimation is subdivided in two steps.

Step 1: Synthesis of the unknown input

The unknown input is partially decoupled based on the nonsingular state and output transformation T_{σ_k} and $U_{\sigma_k}^{-1}$ given repectively by (14) and (19)

$$T_1 = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \\ 0.25 & 0.25 & 0.25 \end{bmatrix}, \; T_2 = \begin{bmatrix} 0 & 1 & 0 \\ -4.73 & 0 & 1 \\ 0.1752 & 0.0871 & 0.1744 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} -0.5 & 1 & 0 \\ -1 & 0 & 1 \\ 0.32 & 0.24 & 0.56 \end{bmatrix}$$

$$U_1 = \begin{bmatrix} -1 & 1 \\ 0.25 & 0.25 \end{bmatrix}, \; U_2 = \begin{bmatrix} -0.9983 & 1 \\ 0.0873 & 0.0871 \end{bmatrix}, \; U_3 = \begin{bmatrix} -0.75 & 1 \\ 0.32 & 0.24 \end{bmatrix}.$$

Step 2: Interval observer design for the unknown input-free subsystem

To relax the design conditions, based on the Yalmip toolbox, we first look for observer gains L_{σ_k} such that (35) holds. Secondly, we propose a nonsingular transformation P_{σ_k} such that the matrices $P_{\sigma_k}(\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k})P_{\sigma_k}^{-1}$ are nonnegative. The parameters in Theorem 2 are given as $\alpha = 0.9$ and $\alpha_1 = 0.1$,

$$M_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \; M_2 = \begin{bmatrix} 0.1036 & 0 \\ 0 & 0.1010 \end{bmatrix}, \; M_3 = \begin{bmatrix} 0.1007 & 0 \\ 0 & 0.1005 \end{bmatrix}.$$

$$S_1 = \begin{bmatrix} -0.035 \\ 0.01 \end{bmatrix}, \; S_2 = \begin{bmatrix} 0.0329 \\ 0.0716 \end{bmatrix}, \; S_3 = \begin{bmatrix} 0.0082 \\ -0.0022 \end{bmatrix}.$$

The interval observer gains are computed by $L_{\sigma_k} = M_{\sigma_k}^{-1}S_{\sigma_k}$ with

$$L_1 = \begin{bmatrix} -0.35 \\ 0.1 \end{bmatrix}, \; L_2 = \begin{bmatrix} 0.3177 \\ 0.7093 \end{bmatrix}, \; L_3 = \begin{bmatrix} 0.0813 \\ -0.022 \end{bmatrix}.$$

The matrices $P_{\sigma_k}, \forall\sigma_k \in \overline{1,3}$ ensuring the nonnegativity of $P_{\sigma_k}(\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k})P_{\sigma_k}^{-1}$ are given by

$$P_1 = \begin{bmatrix} -0.5555 & 0.4444 \\ 0.5555 & 0.5556 \end{bmatrix}, \; P_2 = \begin{bmatrix} 0.0018 & -0.0006 \\ -0.0018 & 1.0006 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0.2109 & 0.1582 \\ -0.2109 & 0.8418 \end{bmatrix}.$$

Furthermore, $\mu = 3.0973$ leads to an average dwell time $\tau_a > 0.4910$. The switched signal σ_k verifying the average dwell time is plotted in Figure 1. The

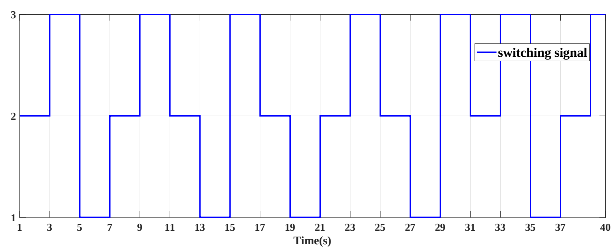

Figure 1: The switching signal

optimal value of γ is equal to 0.1147. Despite the disturbances and the switching instants, the proposed interval observer is depicted in Figure 2 where solid line and dashed lines represent respectively the state and the estimated bounds fulfilling the conditions of Theorem 1 and Theorem 2. This leads to the inclusion $\underline{x} \leq x \leq \overline{x}$ and to the ISS stability of the estimation errors. Moreover, the upper and the lower bound of the unknown input d are drawn in Figure 3 where the solid line is the unknown input. It is enclosed by the dashed lines which are the upper and lower estimated bound. These simulations show the success of the used approach and the robustness with regard to the disturbances.

5. Conclusion

A simultaneous input and state interval observer is proposed in this paper for discrete-time linear switched systems subject to bounded noises and

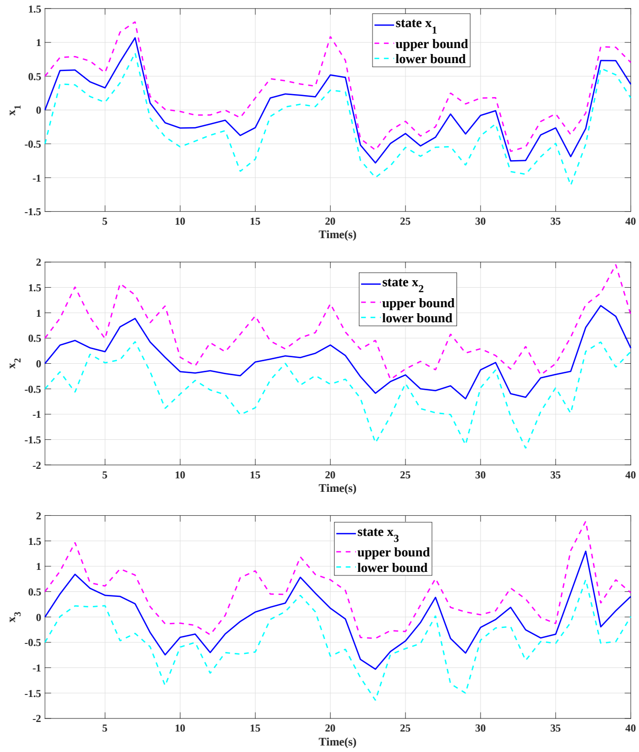

Figure 2: State and estimated bounds

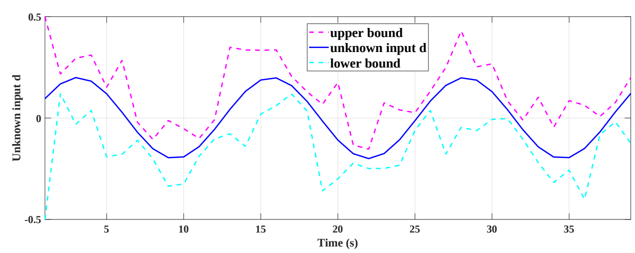

Figure 3: The unknown input

disturbances. Moreover, sufficient conditions for the stability of the interval observer are derived in terms of LMI. Finally, the effectiveness of the proposed approach is shown on a numerical example. For future works, the case of unknown switching signals will be considered. The extension to Linear-Parameter Varying switched systems as well as the application for fault detection are also promising directions to be considered. On the other hand, extension to interval observer design for discrete-time switched systems with unknown inputs in the spirit of what is done in [38] is also an interesting perspective.

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Declaration of interests

☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: