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ASYMPTOTICS FOR SOME DISCRETIZATIONS OF DYNAMICAL SYSTEMS, APPLICATION TO SECOND ORDER SYSTEMS WITH NONLOCAL NONLINEARITIES

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Abstract. In the present paper we study the asymptotic behavior of discretized finite dimensional dynamical systems. We prove that under some discrete angle condition and under a Lojasiewicz's inequality condition, the solutions to an implicit scheme converge to equilibrium points. We also present some numerical simulations suggesting that our results may be extended under weaker assumptions or to infinite dimensional dynamical systems.

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1. Introduction

A question that naturally comes out when considering the study of the asymptotic behavior of dynamical systems is whether noticeable differences arise between continuous models and their discretizations. This is of course of main interests from the mathematical point of view, but it is also very common to introduce continuous models as a limit of a, say, physical model with low or very large scales of unknown interactions involved. In this paper, what we have in mind lies in fact in the discretization of a nonlinear wave equation with a (here nonlocal) nonlinear part for which we give some numerical simulations in part 5.3. Having settled this, by means of an implicit discretization in time, the major results of this paper deal in fact with the asymptotic behavior of a discretized version of the following second order differential equation

$$\ddot{x} + |\dot{x}|^{\alpha}\dot{x} + \nabla F(x) = 0, \tag{1}$$

for some real-valued function F and $\alpha \geq 0$. Different discretizations may be considered. We will here focus on an implicit discretization, in the spirit of the famous paper [1] in which the main results' angular stone is given by the inequality (6) satisfied by a sequence (x_n). For such a sequence, the main results of the present paper, theorems 2.1 and 4.7 state that, on the one hand provided that a Lojasiewicz's inequality is satisfied as well as, on the other hand, the inequality (9) holds, the convergence of this sequence occurs and the speed of convergence is estimated. This inequality (9) finds its origin in a more abstract inequality namely the so-called "angle condition" for continuous systems, which we describe further

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in this section. Though it is not stated this way in [1], the results therein have been considered by the authors in [18] to be more dedicated to explicit Euler schemes, while in [18] an implicit scheme is considered for gradient-systems. The same implicit situation has been independently considered in [4].
In order to illustrate our results, some numerical simulations given in section 5 are presented with assumptions to comply (9), as well as more general situations which incline to think that the results given in section 4 may be extended.
Semi-implicit schemes are also considered in the present paper in section 5 leading also to think that our results may be also extended to this type of scheme.
Concerning the above evoked nonlinear partial differential evolution equation, with a non-local Hölder norm, rewritten in an abstract form equivalent to (1), we also performed some computations. For these, we used an implicit scheme in time together with a finite element method in space (here the space dimension is 1). The results that we obtain therein are in a way consistent with the asymptotic behavior of solutions of (1) as given in [11], [12].

Let us briefly recall about the angle condition. We consider

$$\dot{u}(t) + \mathcal{F}(u(t)) = 0, \quad t \geq 0, \tag{2}$$

where $\mathcal{F} \in C(\mathbb{R}^p; \mathbb{R}^p)$. Let $\mathcal{E} \in C^1(\mathbb{R}^p, \mathbb{R})$. One says that $\mathcal{E}'$ and $\mathcal{F}$ satisfy an angle condition if there exists $\alpha > 0$ such that

$$\langle \mathcal{E}'(u), \mathcal{F}(u) \rangle \geq \alpha \, \|\mathcal{E}'(u)\| \, \|\mathcal{F}(u)\| \text{ for every } u \in \mathbb{R}^p. \tag{3}$$

This condition appeared first in [1] in order to study (2) and discrete systems.
It is clear that (1) can be written in the form (2). Chill et Al. used in [8] this angle condition to study the behavior of solutions of (1) when $\alpha = 0$. More recently, Haraux and Jendoubi generalized in [10] the angle condition by adding a power β to $\|\mathcal{E}'(u)\|$ in order to deal with the equation (1) when $\alpha \neq 0$. In the situation when one studies discretized version of (2), as we already said some "discrete" angle conditions have been considered. To our knowledge, the first one appeared in [1]. To state this condition, one has to consider a sequence (x_n), which will further be a solution to a discretized version of (2) i.e. such that

$$\forall n \in \mathbb{N}, \frac{x_{n+1} - x_n}{\Delta t} + \mathcal{F}(x_n) = 0, \tag{4}$$

or

$$\forall n \in \mathbb{N}, \frac{x_{n+1} - x_n}{\Delta t} + \mathcal{F}(x_{n+1}) = 0, \tag{5}$$

for a given fixed time step $\Delta t > 0$.

A sequence (x_n) is said to satisfied the discrete angle condition given in [1], if the following "explicit" inequality holds :

$$\forall n \in \mathbb{N}, \; \Phi(x_n) - \Phi(x_{n+1}) \geq \sigma \|\nabla \Phi(x_n)\| \|x_{n+1} - x_n\|, \tag{6}$$

where Φ is a real-valued function and $\sigma > 0$. For the purposes of this paper Φ will be related to $\mathcal{F}$. Studying the convergence of the sequence that satisfies (6) or similar inequalities gives then asymptotic behaviors of sequences that are approximated solutions of (2).

It also would be worth to study the situation when

$$\Phi(x_n) - \Phi(x_{n+1}) \geq \sigma \|\nabla\Phi(x_{n+1})\| \|x_{n+1} - x_n\|. \tag{7}$$

To our best knowledge, the question of convergence under this last assumption instead of (6) is open. In [2], Alaa and Pierre studied the convergence under the following assumption

$$\Phi(x_n) - \Phi(x_{n+1}) \geq \sigma[\|\nabla\Phi(x_{n+1})\|^2 + \|x_{n+1} - x_n\|^2]. \tag{8}$$

It is straightforward to see that a sequences satisfying (8) also complies to (7).
In this paper we try to extend the results in [2] by considering the discrete angle condition given by (9).

2. Main results of the paper

Let $\Phi : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a C^1 function, $\sigma > 0$, $\beta \geq 1$ and let us consider a sequence (x_n) satisfying

$$\Phi(x_n) - \Phi(x_{n+1}) \geq \sigma[\|\nabla\Phi(x_{n+1})\|^{\beta+1} + \|x_{n+1} - x_n\|^{\beta+1}]. \tag{9}$$

Theorem 2.1. *We assume that there exists $\theta \in (0, \frac{1}{2}]$ such that*

$$\forall a \in \mathbb{R}^N \ \exists c_a > 0 \ \exists r_a > 0 / \ \forall u \in \mathbb{R}^N : \ \|x - a\| < r_a \Longrightarrow \|\nabla\Phi(x)\| \geq c_a |\Phi(x) - \Phi(a)|^{1-\theta}. \tag{10}$$

Assume also that

$$\beta(1 - \theta) < 1. \tag{11}$$

Let (x_n) be a sequence satisfying (9). Then either $\lim_{n\to+\infty} \|x_n\| = +\infty$, or there exists $x^\infty \in \mathbb{R}^N$ such that $\nabla\Phi(x^\infty) = 0$ and

$$\lim_{n\to+\infty} x_n = x^\infty.$$

Precisely, in this case we have

$$\|x_n - x^\infty\| = \begin{cases} O(e^{-cn}) & \text{for some } c > 0 \text{ if } \beta = \frac{\theta}{1-\theta} \\ O(n^{-\frac{1-\beta(1-\theta)}{\beta(1-\theta)-\theta}}) & \text{if } \beta > \frac{\theta}{1-\theta}. \end{cases} \tag{12}$$

Remark 2.2. Let us note that since $\beta \geq 1$ and $\theta \in (0, \frac{1}{2}]$, then $\beta(1-\theta) \geq \theta$. Let us also remark that the equality $\beta = \frac{\theta}{1-\theta}$ requires $\beta = 1$ and $\theta = \frac{1}{2}$.

Remark 2.3. If there exists $M > 0$ such that $\forall n \in \mathbb{N}$, $\|x_n\| \leq M$, then the assumption (10) may merely apply to those $a \in \mathbb{R}^N$ such that $\|a\| \leq M$.

Remark 2.4. By using Young's inequality, one checks that the sequence (x_n) defined by (9) satisfies

$$\Phi(x_n) - \Phi(x_{n+1}) \geq \sigma\frac{\beta+1}{\beta}\|\nabla\Phi(x_{n+1})\|^\beta \|x_{n+1} - x_n\|. \tag{13}$$

Remark 2.5. Even in the case $\beta = 1$ (see remark 2.7 page 1296 of [2]) counterexamples to the convergence may occur either in the discrete or the continuous case. For the continuous case, one may refer to [17, 14], while for the discrete situation one may refer to [1].

3. Proof of Theorem 2.1

If $\lim\limits_{n\to+\infty} \|x_n\| \neq +\infty$ the sequence (x_n) has a bounded subsequence. Let x^∞ be an accumulation point of the sequence. Since $(\Phi(x_n))$ is a non-increasing sequence, $\lim_{n\to+\infty}\Phi(x_n) = \Phi(x^\infty)$. Assume that for some $n_0 \in \mathbb{N}$, $\Phi(x_{n_0}) = \Phi(x^\infty)$, then for all $n \geq n_0$, $\Phi(x_n) = \Phi(x_{n_0})$. According to (9), $x_n = x_{n_0}$ for all $n \geq n_0$.
Otherwise, for all $n \in \mathbb{N}$, one has $\Phi(x_n) > \Phi(x^\infty)$.
According to (10), we get

$$\exists c_0 > 0\ \exists r > 0/\ \forall x \in \mathbb{R}^n :\ \|x - x^\infty\| < r \Longrightarrow \|\nabla\Phi(x)\| \geq c_0|\Phi(x) - \Phi(x^\infty)|^{1-\theta}. \tag{14}$$

In the sequel, without loss of generality, we assume $\Phi(x^\infty) = 0$.
Since x^∞ is an accumulation point of (x_n), there exists $n_1 \in \mathbb{N}$ such that

$$\left(\frac{1}{\sigma}\right)^{\frac{1}{\beta+1}} [\Phi(x_{n_1})]^{\frac{1}{\beta+1}} < \frac{r}{3}. \tag{15}$$

$$\|x_{n_1} - x^\infty\| < \frac{r}{3}, \quad \text{and} \quad c_1[\Phi(x_{n_1})]^{1-\beta(1-\theta)} + c_2[\Phi(x_{n_1})]^{\frac{1}{\beta+1}} < \frac{r}{3}, \tag{16}$$

where

$$c_1 = \frac{2^{\beta(1-\theta)}\beta}{(1-\beta(1-\theta))\sigma(\beta+1)c_0^\beta}, \qquad c_2 = \frac{1}{\sigma^{\frac{1}{\beta+1}}}\frac{2^{\frac{1}{\beta+1}}}{2^{\frac{1}{\beta+1}} - 1}. \tag{17}$$

Let $K = \sup\{n \geq n_1/\ \forall i \in [\![n_1, n+1]\!],\ \|x_i - x^\infty\| < r\}$, and assume that $K < +\infty$.
Let us note that

$$\forall n \in [\![n_1 - 1, K]\!] \quad \|x_{n+1} - x^\infty\| < r.$$

Let $n \in [\![n_1, K]\!]$. As in [18, 2] we distinguish two cases.
First case, we assume that

$$\Phi(x_{n+1}) > \frac{\Phi(x_n)}{2}. \tag{18}$$

We have

$$\begin{aligned}
& [\Phi(x_n)]^{1-\beta(1-\theta)} - [\Phi(x_{n+1})]^{1-\beta(1-\theta)} \\
= \ & \int_{\Phi(x_{n+1})}^{\Phi(x_n)} (1-\beta(1-\theta))x^{-\beta(1-\theta)}\,dx \\
\geq \ & (1-\beta(1-\theta))[\Phi(x_n) - \Phi(x_{n+1})][\Phi(x_n)]^{-\beta(1-\theta)} \\
\geq \ & 2^{-\beta(1-\theta)}(1-\beta(1-\theta))[\Phi(x_n) - \Phi(x_{n+1})][\Phi(x_{n+1})]^{-\beta(1-\theta)} \qquad \text{(by (18))} \\
\geq \ & 2^{-\beta(1-\theta)}(1-\beta(1-\theta))\sigma\frac{\beta+1}{\beta}\|x_{n+1} - x_n\|\|\nabla\Phi(x_{n+1})\|^\beta[\Phi(x_{n+1})]^{-\beta(1-\theta)} \qquad \text{(by (13))} \\
\geq \ & 2^{-\beta(1-\theta)}(1-\beta(1-\theta))\sigma\frac{\beta+1}{\beta}c_0^\beta\|x_{n+1} - x_n\| \qquad \text{(by (14))}.
\end{aligned}$$

For the second case, we assume that

$$\Phi(x_{n+1}) \leq \frac{\Phi(x_n)}{2}. \tag{19}$$

From (9), we get

$$\begin{aligned}
\|x_{n+1}-x_n\| &\le \frac{1}{\sigma^{\frac{1}{\beta+1}}}[\Phi(x_n)-\Phi(x_{n+1})]^{\frac{1}{\beta+1}}\\
&\le \frac{1}{\sigma^{\frac{1}{\beta+1}}}[\Phi(x_n)]^{\frac{1}{\beta+1}}\\
&\le \frac{1}{\sigma^{\frac{1}{\beta+1}}}\frac{2^{\frac{1}{\beta+1}}}{2^{\frac{1}{\beta+1}}-1}\left([\Phi(x_n)]^{\frac{1}{\beta+1}}-[\Phi(x_{n+1})]^{\frac{1}{\beta+1}}\right) \quad \text{by (19)}
\end{aligned}$$

In both cases we have for all $n\in [\![n_1,K]\!]$

$$\begin{aligned}
&\|x_{n+1}-x_n\|\\
&\le c_1\left([\Phi(x_n)]^{1-\beta(1-\theta)}-[\Phi(x_{n+1})]^{1-\beta(1-\theta)}\right)+c_2\left([\Phi(x_n)]^{\frac{1}{\beta+1}}-[\Phi(x_{n+1})]^{\frac{1}{\beta+1}}\right),
\end{aligned} \tag{20}$$

where c_1 and c_2 are defined in (17).
Now we have

$$\begin{aligned}
&\|x_{K+2}-x_\infty\|\\
\le\ &\|x_{K+2}-x_{K+1}\|+\|x_{K+1}-x_\infty\|\\
\le\ &\left(\frac{1}{\sigma}\right)^{\frac{1}{\beta+1}}[\Phi(x_{K+1})-\Phi(x_{K+2})]^{\frac{1}{\beta+1}}+\|x_{K+1}-x_\infty\| \quad \text{(by (9))}\\
\le\ &\left(\frac{1}{\sigma}\right)^{\frac{1}{\beta+1}}[\Phi(x_{n_1})]^{\frac{1}{\beta+1}}+\|x_{K+1}-x_\infty\|\\
\le\ &\frac{r}{3}+\|x_{K+1}-x_\infty\| \quad \text{(by (15))}\\
\le\ &\frac{r}{3}+\|x_{K+1}-x_{n_1}\|+\|x_{n_1}-x_\infty\|\\
\le\ &\frac{r}{3}+\sum_{k=n_1}^{K}\|x_{k+1}-x_k\|+\frac{r}{3} \quad \text{(by (15))}\\
\le\ &c_1\left([\Phi(x_{n_1})]^{1-\beta(1-\theta)}-[\Phi(x_{K+1})]^{1-\beta(1-\theta)}\right)+c_2\left([\Phi(x_{n_1})]^{\frac{1}{\beta+1}}-[\Phi(x_{K+1})]^{\frac{1}{\beta+1}}\right)+\frac{2r}{3}\\
\le\ &c_1[\Phi(x_{n_1})]^{1-\beta(1-\theta)}+c_2[\Phi(x_{n_1})]^{\frac{1}{\beta+1}}+\frac{2r}{3}\\
<\ &r \quad \text{(by (16))}
\end{aligned}$$

which contradicts the definition of K. Thus $K=\infty$ and (20) is true for all $n\geq n_1$. So $\sum\|x_{n+1}-x_n\|$ converges and so the sequence (x_n).

We now will prove (12). We proved (see (20)) that for all $p\geq n_1$

$$\begin{aligned}
&\|x_{p+1}-x_p\|\\
\le\ &c_1\left([\Phi(x_p)]^{1-\beta(1-\theta)}-[\Phi(x_{p+1})]^{1-\beta(1-\theta)}\right)+c_2\left([\Phi(x_p)]^{\frac{1}{\beta+1}}-[\Phi(x_{p+1})]^{\frac{1}{\beta+1}}\right).
\end{aligned}$$

For any $n \geq n_1$, there holds

$$\begin{aligned} \|x_n - x^\infty\| &\leq \sum_{p=n}^{\infty} \|x_{p+1} - x_p\| \\ &\leq c_1[\Phi(x_n)]^{1-\beta(1-\theta)} + c_2[\Phi(x_n)]^{\frac{1}{\beta+1}} \end{aligned} \tag{21}$$

If there exists $n_2 \geq n_1$ such that $\Phi(x_{n_2}) = 0$, since $(\Phi(x_n))$ is non-increasing, then for all $n \geq n_2$, $\Phi(x_n) = 0$. Using (9), it comes $x_n = x^\infty$ for all $n \geq n_2$. Otherwise, in order to estimate the speed of convergence, we will adopt the related method in [18, 2]. See also [12, 5, 10, 13] where the speed of convergence is given in the continuous case.
Without loss of generality one may assume that $\Phi(x_n) \leq 1$ for all $n \geq n_1$. From (21), it comes

$$\|x_n - x^\infty\| \leq \begin{cases} (c_1 + c_2)[\Phi(x_n)]^{\frac{1}{\beta+1}}, & \text{if } \beta(1-\theta) = \theta, \\ (c_1 + c_2)[\Phi(x_n)]^{1-\beta(1-\theta)}, & \text{if } \beta(1-\theta) > \theta. \end{cases} \tag{22}$$

Note that if $\beta(1-\theta) = \theta$ then $\frac{1}{1+\beta} = 1 - \theta$.
For all $n \geq n_1$, there holds $\|x_n - x^\infty\| < r$, so that according to (14), one has

$$\|\nabla\Phi(x_n)\| \geq c_0 \Phi(x_n)^{1-\theta}. \tag{23}$$

Let

$$\begin{aligned} G : (0, +\infty) &\longrightarrow (0, +\infty) \\ s &\longmapsto G(s) = \begin{cases} -\ln(s) & \text{if } \beta = \frac{\theta}{1-\theta} \\ \frac{1}{[\beta(1-\theta)-\theta]s^{\beta(1-\theta)-\theta}} & \text{if } \beta > \frac{\theta}{1-\theta}. \end{cases} \end{aligned}$$

For all $n \geq n_1$, one has

$$\begin{aligned} G(\Phi(x_{n+1})) - G(\Phi(x_n)) &= \int_{\Phi(x_{n+1})}^{\Phi(x_n)} \frac{ds}{s^{(1-\theta)(1+\beta)}} \\ &\geq \frac{\Phi(x_n) - \Phi(x_{n+1})}{[\Phi(x_n)]^{(1-\theta)(1+\beta)}} \\ &\geq \frac{\sigma\|\nabla\Phi(x_n)\|^{1+\beta}}{[\Phi(x_n)]^{(1-\theta)(1+\beta)}} \quad \text{(by (9))} \\ &\geq c_0^{1+\beta}\sigma \quad \text{(by (23))}. \end{aligned}$$

We therefore get

$$\forall n \geq n_1 \quad G(\Phi(x_n)) - G(\Phi(x_{n_1})) \geq c_0^{1+\beta}\sigma(n - n_1). \tag{24}$$

Now if we assume that $\beta = \frac{\theta}{1-\theta}$, we get

$$\forall n \geq n_1 \quad -\ln(\Phi(x_n)) + \ln(\Phi(x_{n_1})) \geq c_0^{1+\beta}\sigma(n - n_1),$$

or

$$\forall n \geq n_1 \quad \Phi(x_n) \leq (\Phi(x_{n_1}))e^{-c_0^{1+\beta}\sigma(n-n_1)}.$$

Thus, from (22), it comes

$$\forall n \geq n_1 \quad \|x_n - x^\infty\| \leq [\Phi(x_{n_1})]^{1-\theta} e^{-c_0^{1+\beta}\sigma(1-\theta)(n-n_1)}.$$

Likewise if $\beta > \frac{\theta}{1-\theta}$, we infer from (24)

$$\forall n \geq n_1 \quad G(\Phi(x_n)) \geq c_0^{1+\beta}\sigma(n-n_1) + G(\Phi(x_{n_1})).$$

So for all $n \geq n_1$

$$(\beta(1-\theta)-\theta)[\Phi(x_n)]^{\beta(1-\theta)-\theta} \leq \frac{1}{c_0^{1+\beta}\sigma(n-n_1) + G(\Phi(x_{n_1}))}$$

or

$$[\Phi(x_n) - \Phi(x^\infty)] \leq \left(\frac{1}{\beta(1-\theta)-\theta}\frac{1}{c_0^{1+\beta}\sigma(n-n_1) + G(\Phi(x_{n_1}))}\right)^{\frac{1}{\beta(1-\theta)-\theta}}$$

(12) follows again using(22).

4. Application to an implicit scheme

In this section we apply our method to the following example, which to our best knowledge, has not been considered yet in our the study of asymptotic behaviors.
We consider a sequence $(u_n, v_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$\left\{\begin{aligned} \frac{u_{n+1}-u_n}{\Delta t} &= v_{n+1} \\ \frac{v_{n+1}-v_n}{\Delta t} &= -\|v_{n+1}\|^\alpha v_{n+1} - \nabla F(u_{n+1}) \\ u_0,\ v_0 &\in \mathbb{R}^d \end{aligned}\right. \tag{25}$$

where $\alpha > 0$ and $F : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a C^1 function such that ($\langle .,\rangle$ denoting the standard scalar product on $\mathbb{R}^d$)

$$\exists c_F > 0/\ \forall u, v \in \mathbb{R}^d \quad \langle \nabla F(u) - \nabla F(v), u - v\rangle \geq -c_F\|u-v\|^{\alpha+2}, \tag{26}$$

$$\exists L_F > 0/\ \forall u, v \in \mathbb{R}^d \quad \|\nabla F(u) - \nabla F(v)\| \leq L_F\|u-v\|. \tag{27}$$

Let us remark that this case is more general than the case when $\alpha = 0$ already treated in [9], we think that our example thus generalizes the examples therein, though we require somewhat strong assumptions in order to have an ascertained situation.

Remark 4.1. Let us remark that the assumption that, for C^2 functions, (27) is equivalent to the fact that $\nabla^2 F$ is bounded, while condition (26) implies that F is convex. Since, the condition (26) is only used in the proof of the next proposition and lemma 4.4, future works examining situations when (26) is not satisfied would probably be of nice interests.

The existence and uniqueness of a sequence satisfying (25) is not clear in general. The proposition below gives some sufficient conditions for which it is the case.

Proposition 4.2. *Assume that F is of class $C^1(\mathbb{R}^d)$, coercive and that* (26) *and* (27) *hold, then for any $(u_0, v_0) \in \mathbb{R}^{2d}$, provided Δt is small enough, the sequence (u_n, v_n) given by* (25) *is well defined, and we have*

$$\forall n \in \mathbb{N} \quad \frac{1}{2}\|v_{n+1}\|^2 + F(u_{n+1}) \le \frac{1}{2}\|v_n\|^2 + F(u_n). \tag{28}$$

Proof. Let us now denote $E_0 = \frac{1}{2}\|v_0\|^2 + F(u_0)$. Due to the coercivity of F, we can choose some $R > 0$ such that if $F(u) \le E_0$ then $u \in B(0, R)$. Let us also denote $R' = \sqrt{2E_0}$.
Let us consider some function $N_1 : \mathbb{R}^d \to \mathbb{R}^d$ which is continuous and bounded and globally lipschitz on $\mathbb{R}^d$ and such that $N_1(v) := \|v\|^\alpha v$ on $B(0, 3R')$. Let us also consider a function $N_2 : \mathbb{R}^d \to \mathbb{R}^d$ which is continuous and bounded and globally lipschitz and such that $N_2 = -\nabla F$ on $B(0, 3R)$. It is standard (see [6]) that provided Δt is small enough, there exists a unique sequence (u_n, v_n) such that

$$\begin{cases} \dfrac{u_{n+1} - u_n}{\Delta t} = v_{n+1} \\ \dfrac{v_{n+1} - v_n}{\Delta t} = -N_1(v_{n+1}) + N_2(u_{n+1}). \end{cases} \tag{29}$$

Let us remark that we have

$$\|u_1\| \le \|u_0\| + \Delta t\|v_1\|$$
$$\|v_1\| \le \|v_0\| + M\Delta t$$

where $M = \|N_1\|_\infty + \|N_2\|_\infty$.
Thus we get

$$\|u_1\| \le \|u_0\| + \Delta t(\|v_0\| + M\Delta t)$$
$$\|v_1\| \le \|v_0\| + M\Delta t.$$

Thus for Δt small enough depending only on R and R' we have

$$(u_1, v_1) \in B(0, 2R) \times B(0, 2R')$$

and thus (u_1, v_1) satisfies

$$\begin{cases} \dfrac{u_1 - u_0}{\Delta t} = v_1 \\ \dfrac{v_1 - v_0}{\Delta t} = -\|v_1\|^\alpha v_1 - \nabla F(u_1) \end{cases} . \tag{30}$$

Now if we apply the proof of Proposition 4.5 below with $n = 0$ we get that if Δt is small enough we have (28) with $n = 0$ (for $n = 0$ the proof of Proposition 4.5 only requires that assumptions (26) and (27) are satisfied and the fact that (u_1, v_1) are solutions of (30)). We thus get $F(u_1) \le E_0$ and $\|v_1\| \le R'$, we can thus proceed by induction and this gives the existence and uniqueness of a sequence satisfying (25). □

In the sequel all the results can be stated under the existence and uniqueness of a sequence satisfying (29), existence and uniqueness that we will assume. The previous proposition gives a sufficient condition for which this assumption is true.

Let $\mathcal{S} = \{a \in \mathbb{R}^d \ / \ \nabla F(a) = 0\}$. We assume also that there exists $\theta \in (0, \frac{1}{2}]$ such that

$$\forall a \in \mathcal{S} \ \exists \delta_a > 0 \ \exists \nu_a > 0 / \ \forall u \in \mathbb{R}^d : \ \|u - a\| < \delta_a \Longrightarrow \|\nabla F(u)\| \geq \nu_a |F(u) - F(a)|^{1-\theta}. \tag{31}$$

Proposition 4.3. *([15, 16, 7, 5]) Assumption (31) is satisfied if one of the following two cases holds:*
- F is a polynomial, or
- F is analytic and $\mathcal{S}$ is compact.

The proof is given in the appendix.

Lemma 4.4. *The assumption (26) on F implies that*

$$\forall u, v \in \mathbb{R}^d \quad F(v) \geq F(u) + \langle \nabla F(u), v - u \rangle - \frac{c_F}{2} \|u - v\|^{\alpha+2}.$$

The proof is given in the appendix.

The energy of the system is defined by

$$E(u, v) = \frac{1}{2}\|v\|^2 + F(u).$$

Proposition 4.5. *Assume F satisfies (26). Let (u_n, v_n) be a sequence satisfying (25), then we have*

$$E(u_{n+1}, v_{n+1}) - E(u_n, v_n) \leq -\Delta t \left[1 - \frac{c_F}{2}(\Delta t)^{\alpha+1}\right] \|v_{n+1}\|^{\alpha+2}.$$

Proof. By taking the scalar product of the second relation of (25) with $\Delta t v_{n+1}$, it comes

$$\langle \frac{v_{n+1} - v_n}{\Delta t}, \Delta t v_{n+1} \rangle = -\Delta t \|v_{n+1}\|^{\alpha+2} - \langle \nabla F(u_{n+1}), \Delta t v_{n+1} \rangle$$

or

$$\|v_{n+1}\|^2 - \langle v_n, v_{n+1} \rangle = -\Delta t \|v_{n+1}\|^{\alpha+2} - \langle \nabla F(u_{n+1}), u_{n+1} - u_n \rangle. \tag{32}$$

By using the Cauchy-Schwarz inequality, there holds

$$-\langle v_n, v_{n+1} \rangle \geq -\frac{1}{2}\|v_{n+1}\|^2 - \frac{1}{2}\|v_n\|^2. \tag{33}$$

Combining (32) and (33), if follows that

$$\frac{1}{2}\|v_{n+1}\|^2 - \frac{1}{2}\|v_n\|^2 \leq -\Delta t \|v_{n+1}\|^{\alpha+2} - \langle \nabla F(u_{n+1}), u_{n+1} - u_n \rangle,$$

and then

$$E(u_{n+1}, v_{n+1}) - E(u_n, v_n) \leq -\Delta t \|v_{n+1}\|^{\alpha+2} + F(u_{n+1}) - F(u_n) - \langle \nabla F(u_{n+1}), u_{n+1} - u_n \rangle.$$

By using lemma 4.4, we get

$$\begin{aligned} E(u_{n+1}, v_{n+1}) - E(u_n, v_n) &\leq -\Delta t \|v_{n+1}\|^{\alpha+2} + \frac{c_F}{2}\|u_{n+1} - u_n\|^{\alpha+2} \\ &= -\Delta t \left[1 - \frac{c_F}{2}(\Delta t)^{\alpha+1}\right] \|v_{n+1}\|^{\alpha+2}, \end{aligned}$$

from which the proof is completed. □

In order to study the asymptotics of the sequence (u_n), we define the $\omega-$limit set

$$\omega((u_n)_{n\in\mathbb{N}}) = \{a \in \mathbb{R}^d : \exists n_k \to \infty / u_{n_k} \longrightarrow a\}.$$

Corollary 4.6. *Let F satisfying (26) and assume that $0 < \Delta t < \left(\frac{2}{c_F}\right)^{\frac{1}{\alpha+1}}$. Let (u_n, v_n) be a sequence satisfying (25). If (u_n) is bounded, then $\lim_{n\to\infty} E(u_n, v_n)$ exists, $v_n \longrightarrow 0$ and $\omega((u_n)_{n\in\mathbb{N}})$ is a nonempty compact connected subset of $\mathcal{S}$.*

Proof. According to proposition 4.5, $(E(u_n, v_n))$ is non increasing, thus converges in $\mathbb{R} \cup \{-\infty\}$. From the boundedness of (u_n), we deduce that $(E(u_n, v_n))$ converges to a real number. Using again proposition 4.5, $\sum \|v_{n+1}\|^{\alpha+2}$ converges, so (v_n) tends to 0. Now as (u_n) is bounded, the set $\omega((u_n)_{n\in\mathbb{N}})$ is compact in $\mathbb{R}^d$. Besides, from the first relation of (25), one deduces that $(u_{n+1} - u_n)$ tends to 0 . It is standard to prove that $\omega((u_n)_{n\in\mathbb{N}})$ is a connected part of $\mathbb{R}^d$. The second relation in (25) shows that $\omega((u_n)_{n\in\mathbb{N}}) \subset \mathcal{S}$. □

Theorem 4.7. *Let $F : \mathbb{R}^d \longrightarrow \mathbb{R}$ C^2 satisfying (26), (27) and (31). Assume also that $0 < \Delta t < \left(\frac{2}{c_F}\right)^{\frac{1}{\alpha+1}}$ and $\alpha \in (0, \frac{\theta}{1-\theta})$. Let (u_n, v_n) be a sequence satisfying (25) and we assume that (u_n) is bounded. Then there exists $a \in \mathcal{S}$ such that*

$$\lim_{n\to+\infty} \|v_n\| + \|u_n - a\| = 0.$$

In addition as $n \to +\infty$ we have

$$\|u_n - a\| = O\left(n^{-\frac{\theta-(1-\theta)\alpha}{1-2\theta+\alpha(1-\theta)}}\right) \tag{34}$$

Remark 4.8. If F is coercive (i.e. $\lim_{\|u\|\to\infty} F(u) = +\infty$), then (u_n) is bounded.

Remark 4.9. The case $\theta = \frac{\alpha+1}{\alpha+2}$ never occurs since $\theta \in (0, \frac{1}{2}]$ and $\frac{\alpha+1}{\alpha+2} > \frac{1}{2}$.

Proof. By corollary 4.6, $v_n \longrightarrow 0$ and $\omega((u_n)_{n\in\mathbb{N}})$ is nonempty. Let $a \in \omega((u_n)_{n\in\mathbb{N}})$. Then there exists $n_k \to \infty$ such that $u_{n_k} \longrightarrow a$. By continuity of E, we have $\lim_{n\to\infty} E(u_n, v_n) = F(a)$. Up to make the variable change $u = a + w$ and if we set $g(w) = F(a + u) - F(a)$ (hence $\nabla g(w) = \nabla F(u)$), we can assume that $a = 0$ and then $F(0) = 0$, $\nabla F(0) = 0$.

Now let ε be a positive real, and we define for all $u, v \in \mathbb{R}^d$

$$\Phi_\varepsilon(u, v) = E(u, v) + \varepsilon\|\nabla F(u)\|^\alpha\langle\nabla F(u), v\rangle.$$

Let us define $x_n = (u_n, v_n)$. According to the proposition 4.5, for all $n \in \mathbb{N}$:

$$\begin{aligned}
&\Phi_\varepsilon(x_{n+1}) - \Phi_\varepsilon(x_n) \\
\leq\ & -\Delta t\left[1 - \frac{c_F}{2}(\Delta t)^{\alpha+1}\right]\|v_{n+1}\|^{\alpha+2} + \varepsilon[\underbrace{\|\nabla F(u_{n+1})\|^\alpha\langle\nabla F(u_{n+1}), v_{n+1}\rangle}_{T_1} \\
& \underbrace{-\|\nabla F(u_n)\|^\alpha\langle\nabla F(u_n), v_n\rangle}_{T_2}]
\end{aligned}$$

$$
\begin{aligned}
T_1 &= \|\nabla F(u_{n+1})\|^{\alpha}\langle \nabla F(u_{n+1}), v_n - \Delta t\|v_{n+1}\|^{\alpha} v_{n+1} - \Delta t \nabla F(u_{n+1})\rangle \\
&= -\Delta t\|\nabla F(u_{n+1})\|^{\alpha+2} - \Delta t\|\nabla F(u_{n+1})\|^{\alpha}\|v_{n+1}\|^{\alpha}\langle \nabla F(u_{n+1}), v_{n+1}\rangle + \\
&\qquad +\|\nabla F(u_{n+1})\|^{\alpha}\langle \nabla F(u_{n+1}), v_n\rangle \\
&\leq -\Delta t\|\nabla F(u_{n+1})\|^{\alpha+2} + \Delta t\|\nabla F(u_{n+1})\|^{\alpha+1}\|v_{n+1}\|^{\alpha+1} \\
&\qquad +\|\nabla F(u_{n+1})\|^{\alpha}\langle \nabla F(u_{n+1}), v_n\rangle
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
T_2 &= -\|\nabla F(u_n)\|^{\alpha}\langle \nabla F(u_n), v_n\rangle \\
&= -\|\nabla F(u_n)\|^{\alpha}\langle \nabla F(u_n) - \nabla F(u_{n+1}) + \nabla F(u_{n+1}), v_n\rangle \\
&= -\|\nabla F(u_n)\|^{\alpha}\langle \nabla F(u_n) - \nabla F(u_{n+1}), v_n\rangle - \|\nabla F(u_n)\|^{\alpha}\langle \nabla F(u_{n+1}), v_n\rangle \\
&\leq \|\nabla F(u_n)\|^{\alpha}\|\nabla F(u_n) - \nabla F(u_{n+1})\|\|v_n\| - \|\nabla F(u_n)\|^{\alpha}\langle \nabla F(u_{n+1}), v_n\rangle \\
&\leq L_F\Delta t\|\nabla F(u_n)\|^{\alpha}\|v_{n+1}\|\|v_n\| - \|\nabla F(u_n)\|^{\alpha}\langle \nabla F(u_{n+1}), v_n\rangle.
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
& L_F\Delta t\|\nabla F(u_n)\|^{\alpha}\|v_{n+1}\|\|v_n\| \\
\leq\ & L_F\Delta t\|\nabla F(u_n) - \nabla F(u_{n+1}) + \nabla F(u_{n+1})\|^{\alpha}\|v_{n+1}\|\|v_n\| \\
\leq\ & L_F\Delta t\left[\|\nabla F(u_n) - \nabla F(u_{n+1})\|^{\alpha} + \|\nabla F(u_{n+1})\|^{\alpha}\right]\|v_{n+1}\|\|v_n\| \\
\leq\ & L_F\Delta t\left[(L_F\Delta t)^{\alpha}\|v_{n+1}\|^{\alpha} + \|\nabla F(u_{n+1})\|^{\alpha}\right]\|v_{n+1}\|\|v_n\| \quad (\text{ by (27) and (25)}) \\
\leq\ & (L_F\Delta t)^{\alpha+1}\|v_{n+1}\|^{\alpha+1}\|v_n\| + L_F\Delta t\|\nabla F(u_{n+1})\|^{\alpha}\|v_{n+1}\|\|v_n\| \\
\leq\ & (L_F\Delta t)^{\alpha+1}\|v_{n+1}\|^{\alpha+1}\|v_{n+1} + \Delta t\|v_{n+1}\|^{\alpha}v_{n+1} + \Delta t\nabla F(u_{n+1})\| \\
& \quad +L_F\Delta t\|\nabla F(u_{n+1})\|^{\alpha}\|v_{n+1}\|\|v_{n+1} + \Delta t\|v_{n+1}\|^{\alpha}v_{n+1} + \Delta t\nabla F(u_{n+1})\| \quad (\text{ by (25)}) \\
\leq\ & (L_F\Delta t)^{\alpha+1}\|v_{n+1}\|^{\alpha+2} + L_F^{\alpha+1}(\Delta t)^{\alpha+2}\|v_{n+1}\|^{2\alpha+2} + L_F^{\alpha+1}(\Delta t)^{\alpha+2}\|v_{n+1}\|^{\alpha+1}\|\nabla F(u_{n+1})\| \\
& \quad +L_F\Delta t\|\nabla F(u_{n+1})\|^{\alpha}\|v_{n+1}\|^2 + L_F(\Delta t)^2\|\nabla F(u_{n+1})\|^{\alpha}\|v_{n+1}\|^{\alpha+2} \\
& \quad +L_F(\Delta t)^2\|\nabla F(u_{n+1})\|^{\alpha+1}\|v_{n+1}\|.
\end{aligned}
\tag{37}
$$

Since $v_n \longrightarrow 0$, then without loss of generality we assume that for all

$$
\forall n \in \mathbb{N}, \quad \|v_n\| \leq 1. \tag{38}
$$

From (35) we get

$$
\begin{aligned}
T_1 \leq\ & -\Delta t\|\nabla F(u_{n+1})\|^{\alpha+2} + \Delta t\|\nabla F(u_{n+1})\|^{\alpha+1}\|v_{n+1}\| \\
& +\|\nabla F(u_{n+1})\|^{\alpha}\langle \nabla F(u_{n+1}), v_n\rangle.
\end{aligned}
\tag{39}
$$

Also from (37) we get

$$
\begin{aligned}
& L_F\Delta t\|\nabla F(u_n)\|^{\alpha}\|v_{n+1}\|\|v_n\| \\
\leq\ & (L_F\Delta t)^{\alpha+1}\|v_{n+1}\|^{\alpha+2} + L_F^{\alpha+1}(\Delta t)^{\alpha+2}\|v_{n+1}\|^{\alpha+2} \\
+\ & L_F^{\alpha+1}(\Delta t)^{\alpha+2}\|v_{n+1}\|^{\alpha+1}\|\nabla F(u_{n+1})\| \\
& +L_F\Delta t\|\nabla F(u_{n+1})\|^{\alpha}\|v_{n+1}\|^2 + L_F(\Delta t)^2\|\nabla F(u_{n+1})\|^{\alpha}\|v_{n+1}\|^2 \\
& +L_F(\Delta t)^2\|\nabla F(u_{n+1})\|^{\alpha+1}\|v_{n+1}\| \\
\leq\ & (L_F\Delta t)^{\alpha+1}(1+\Delta t)\|v_{n+1}\|^{\alpha+2} + L_F^{\alpha+1}(\Delta t)^{\alpha+2}\|v_{n+1}\|^{\alpha+1}\|\nabla F(u_{n+1})\| \\
& \quad +L_F\Delta t(1+\Delta t)\|\nabla F(u_{n+1})\|^{\alpha}\|v_{n+1}\|^2 + L_F(\Delta t)^2\|\nabla F(u_{n+1})\|^{\alpha+1}\|v_{n+1}\|.
\end{aligned}
\tag{40}
$$

Using (40) in (36), we get

$$T_2 \le (L_F\Delta t)^{\alpha+1}(1+\Delta t)\|v_{n+1}\|^{\alpha+2} + L_F^{\alpha+1}(\Delta t)^{\alpha+2}\|v_{n+1}\|^{\alpha+1}\|\nabla F(u_{n+1})\| + L_F\Delta t(1+\Delta t)\|\nabla F(u_{n+1})\|^{\alpha}\|v_{n+1}\|^2 + L_F(\Delta t)^2\|\nabla F(u_{n+1})\|^{\alpha+1}\|v_{n+1}\| - \|\nabla F(u_n)\|^{\alpha}\langle\nabla F(u_{n+1}), v_n\rangle. \tag{41}$$

On the other hand

$$\begin{aligned}
& \|\nabla F(u_{n+1})\|^{\alpha}\langle\nabla F(u_{n+1}), v_n\rangle - \|\nabla F(u_n)\|^{\alpha}\langle\nabla F(u_{n+1}), v_n\rangle \\
= & [\|\nabla F(u_{n+1})\|^{\alpha} - \|\nabla F(u_n)\|^{\alpha}]\langle\nabla F(u_{n+1}), v_n\rangle \\
\le & |\|\nabla F(u_{n+1})\|^{\alpha} - \|\nabla F(u_n)\|^{\alpha}|\|\nabla F(u_{n+1})\|\|v_n\| \\
\le & \|\nabla F(u_{n+1}) - \nabla F(u_n)\|^{\alpha}\|\nabla F(u_{n+1})\|\|v_n\| \\
\le & (L_F\Delta t)^{\alpha}\|v_{n+1}\|^{\alpha}\|\nabla F(u_{n+1})\|\|v_n\| \quad (\text{ by (27) and (25)}) \\
\le & (L_F\Delta t)^{\alpha}\|v_{n+1}\|^{\alpha}\|\nabla F(u_{n+1})\|\|v_{n+1} + \Delta t\|v_{n+1}\|^{\alpha}v_{n+1} + \Delta t\nabla F(u_{n+1})\| \quad (\text{ by (25)}) \\
\le & (L_F\Delta t)^{\alpha}\|v_{n+1}\|^{\alpha+1}\|\nabla F(u_{n+1})\| + (L_F)^{\alpha}(\Delta t)^{\alpha+1}\|v_{n+1}\|^{2\alpha+1}\|\nabla F(u_{n+1})\| + \\
& \quad (L_F)^{\alpha}(\Delta t)^{\alpha+1}\|v_{n+1}\|^{\alpha}\|\nabla F(u_{n+1})\|^2 \\
\le & (L_F\Delta t)^{\alpha}(1+\Delta t)\|v_{n+1}\|^{\alpha+1}\|\nabla F(u_{n+1})\| + (L_F)^{\alpha}(\Delta t)^{\alpha+1}\|v_{n+1}\|^{\alpha}\|\nabla F(u_{n+1})\|^2. \qquad (42)
\end{aligned}$$

By using (39) (41) and (42) we obtain

$$\begin{aligned}
T_1 + T_2 \le & -\Delta t\|\nabla F(u_{n+1})\|^{\alpha+2} + \Delta t\|\nabla F(u_{n+1})\|^{\alpha+1}\|v_{n+1}\| \\
& + (L_F\Delta t)^{\alpha+1}(1+\Delta t)\|v_{n+1}\|^{\alpha+2} + L_F^{\alpha+1}(\Delta t)^{\alpha+2}\|v_{n+1}\|^{\alpha+1}\|\nabla F(u_{n+1})\| \\
& + L_F\Delta t(1+\Delta t)\|\nabla F(u_{n+1})\|^{\alpha}\|v_{n+1}\|^2 + L_F(\Delta t)^2\|\nabla F(u_{n+1})\|^{\alpha+1}\|v_{n+1}\| \\
& + \|\nabla F(u_{n+1})\|^{\alpha}\langle\nabla F(u_{n+1}), v_n\rangle - \|\nabla F(u_n)\|^{\alpha}\langle\nabla F(u_{n+1}), v_n\rangle \\
\le & -\Delta t\|\nabla F(u_{n+1})\|^{\alpha+2} + \Delta t\|\nabla F(u_{n+1})\|^{\alpha+1}\|v_{n+1}\| + (L_F\Delta t)^{\alpha+1}(1+\Delta t)\|v_{n+1}\|^{\alpha+2} \\
& + L_F^{\alpha+1}(\Delta t)^{\alpha+2}\|v_{n+1}\|^{\alpha+1}\|\nabla F(u_{n+1})\| + L_F\Delta t(1+\Delta t)\|\nabla F(u_{n+1})\|^{\alpha}\|v_{n+1}\|^2 \\
& + L_F(\Delta t)^2\|\nabla F(u_{n+1})\|^{\alpha+1}\|v_{n+1}\| + (L_F\Delta t)^{\alpha}(1+\Delta t)\|v_{n+1}\|^{\alpha+1}\|\nabla F(u_{n+1})\| \\
& + (L_F)^{\alpha}(\Delta t)^{\alpha+1}\|v_{n+1}\|^{\alpha}\|\nabla F(u_{n+1})\|^2 \\
\le & -\Delta t\|\nabla F(u_{n+1})\|^{\alpha+2} + \Delta t(1+L_F\Delta t)\|\nabla F(u_{n+1})\|^{\alpha+1}\|v_{n+1}\| \\
& + (L_F\Delta t)^{\alpha+1}(1+\Delta t)\|v_{n+1}\|^{\alpha+2} \\
& + \left[L_F^{\alpha+1}(\Delta t)^{\alpha+2} + (L_F\Delta t)^{\alpha}(1+\Delta t)\right]\|v_{n+1}\|^{\alpha+1}\|\nabla F(u_{n+1})\| \\
& + L_F\Delta t(1+\Delta t)\|\nabla F(u_{n+1})\|^{\alpha}\|v_{n+1}\|^2 + (L_F)^{\alpha}(\Delta t)^{\alpha+1}\|v_{n+1}\|^{\alpha}\|\nabla F(u_{n+1})\|^2.
\end{aligned}$$

Using now Young's inequality, we find some constants $c_3, c_4, c_5, c_6 > 0$ such that

$$\Delta t(1+L_F\Delta t)\|\nabla F(u_{n+1})\|^{\alpha+1}\|v_{n+1}\| \le \frac{\Delta t}{5}\|\nabla F(u_{n+1})\|^{\alpha+2} + c_3\|v_{n+1}\|^{\alpha+2}$$

$$\left[L_F^{\alpha+1}(\Delta t)^{\alpha+2} + (L_F\Delta t)^{\alpha}(1+\Delta t)\right]\|v_{n+1}\|^{\alpha+1}\|\nabla F(u_{n+1})\| \le \frac{\Delta t}{5}\|\nabla F(u_{n+1})\|^{\alpha+2} + c_4\|v_{n+1}\|^{\alpha+2}$$

$$L_F\Delta t(1+\Delta t)\|\nabla F(u_{n+1})\|^{\alpha}\|v_{n+1}\|^2 \le \frac{\Delta t}{5}\|\nabla F(u_{n+1})\|^{\alpha+2} + c_5\|v_{n+1}\|^{\alpha+2}$$

$$(L_F)^{\alpha}(\Delta t)^{\alpha+1}\|v_{n+1}\|^{\alpha}\|\nabla F(u_{n+1})\|^2 \le \frac{\Delta t}{5}\|\nabla F(u_{n+1})\|^{\alpha+2} + c_6\|v_{n+1}\|^{\alpha+2}.$$

Hence

$$T_1 + T_2 \leq -\frac{\Delta t}{5}\|\nabla F(u_{n+1})\|^{\alpha+2} + \left((L_F\Delta t)^{\alpha+1}(1+\Delta t) + c_3 + c_4 + c_5 + c_6\right)\|v_{n+1}\|^{\alpha+2}. \tag{43}$$

Then

$$\begin{aligned}\Phi_\varepsilon(x_{n+1}) - \Phi_\varepsilon(x_n) &\leq -\Delta t\left[1 - \frac{c_F}{2}(\Delta t)^{\alpha+1}\right]\|v_{n+1}\|^{\alpha+2} + \varepsilon(T_1+T_2)\\ &\leq -\Delta t\left[1 - \frac{c_F}{2}(\Delta t)^{\alpha+1}\right]\|v_{n+1}\|^{\alpha+2} - \varepsilon\left[\frac{\Delta t}{5}\right]\|\nabla F(u_{n+1})\|^{\alpha+2}\\ &\quad +\varepsilon\left((L_F\Delta t)^{\alpha+1}(1+\Delta t) + c_3 + c_4 + c_5 + c_6\right)\|v_{n+1}\|^{\alpha+2}.\end{aligned}$$

By choosing $\varepsilon = \overline{\varepsilon} > 0$ small enough, we get constants $\gamma, \gamma' > 0$ such that

$$\begin{aligned}\Phi_{\overline{\varepsilon}}(x_n) - \Phi_{\overline{\varepsilon}}(x_{n+1}) &\geq \gamma'\left[\|v_{n+1}\|^{\alpha+2} + \|\nabla F(u_{n+1})\|^{\alpha+2}\right]\\ &\geq \gamma\left[\|v_{n+1}\| + \|\nabla F(u_{n+1})\|\right]^{\alpha+2}.\end{aligned} \tag{44}$$

Let us show that when (u_n) is a bounded sequence, (x_n) satisfies (9) with the function $\Phi_{\overline{\varepsilon}}$. Indeed, a simple computation gives

$$\begin{aligned}&\nabla\Phi_{\overline{\varepsilon}}(u,v)\\ =\;&\begin{pmatrix}\nabla F(u) + \overline{\varepsilon}\|\nabla F(u)\|^\alpha\nabla^2F(u)\cdot v + \overline{\varepsilon}\alpha\|\nabla F(u)\|^{\alpha-2}\langle\nabla F(u), v\rangle\nabla^2F(u)\cdot\nabla F(u)\\ v + \overline{\varepsilon}\|\nabla F(u)\|^\alpha\nabla F(u)\end{pmatrix}.\end{aligned}$$

For it is assumed that (u_n) is bounded and that $\|v_n\| \leq 1$, there exists a constant $\eta > 0$ such that

$$\forall n\in\mathbb{N}\quad \|\nabla\Phi_{\overline{\varepsilon}}(x_{n+1})\| \leq \eta[\|v_{n+1}\| + \|\nabla F(u_{n+1})\|]. \tag{45}$$

On the other hand

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(u_{n+1}-u_n, v_{n+1}-v_n)\|\\ &= \|(\Delta t v_{n+1}, -\Delta t\|v_{n+1}\|^\alpha v_{n+1} - \Delta t\nabla F(u_{n+1}))\|\\ &\leq 2\left(\frac{2}{c_F}\right)^{\frac{1}{\alpha+1}}[\|v_{n+1}\| + \|\nabla F(u_{n+1})\|].\end{aligned} \tag{46}$$

By combining (44) – (45) and (46), we get that $(\Phi_{\overline{\varepsilon}}(x_n))$ satisfies (9) with $\beta = \alpha+1$ and $\sigma = \min\left(\frac{\gamma\, c_F}{4}\left(\frac{c_F}{2}\right)^{\frac{1}{\alpha+1}}\frac{1}{2^{\alpha+2}}, \frac{\gamma}{2\eta^{\alpha+2}}\right)$.

From the remark 2.3, let $B \subset \mathbb{R}^d\times\mathbb{R}^d$ be a ball containing (u_n, v_n). For all $(u,v)\in B$

$$\begin{aligned}&\|\nabla\Phi_{\overline{\varepsilon}}(u,v)\|\\ =\;&\|\nabla F(u) + \overline{\varepsilon}\|\nabla F(u)\|^\alpha\nabla^2F(u)\cdot v + \overline{\varepsilon}\alpha\|\nabla F(u)\|^{\alpha-2}\langle\nabla F(u),v\rangle\nabla^2F(u)\cdot\nabla F(u)\| +\\ &\quad\|v + \overline{\varepsilon}\|\nabla F(u)\|^\alpha\nabla F(u)\|\\ =\;&\|\nabla F(u)\| - \overline{\varepsilon}\|\|\nabla F(u)\|^\alpha\nabla^2F(u)\cdot v + \overline{\varepsilon}\alpha\|\nabla F(u)\|^{\alpha-2}\langle\nabla F(u),v\rangle\nabla^2F(u)\cdot\nabla F(u)\| +\\ &\quad\|v\| - \overline{\varepsilon}\|\|\nabla F(u)\|^\alpha\nabla F(u)\|\\ \geq\;&(1-\overline{\varepsilon}C)[\|v\| + \|\nabla F(u)\|].\end{aligned}$$

By possibly taking $\overline{\varepsilon} > 0$ smaller, there exists $\rho > 0$ such that

$$\forall (u,v) \in B \qquad \|\nabla \Phi_{\overline{\varepsilon}}(u,v)\| \geq \rho[\|v\| + \|\nabla F(u)\|]. \tag{47}$$

If (a,b) is not a critical point of $\Phi_{\overline{\varepsilon}}$, then $\Phi_{\overline{\varepsilon}}$ satisfies (10) with $\theta = \frac{1}{2}$ as best exponent, thanks to the continuity of $\Phi_{\overline{\varepsilon}}$.
Let $(a,b) \in B$ be a critical point of $\Phi_{\overline{\varepsilon}}$. Then $\nabla F(a) = 0$ and $b = 0$. From (31)

$$\exists \delta_a > 0 \; \exists \nu_a > 0 / \; \forall u \in \mathbb{R}^d : \; \|u - a\| < \delta_a \Longrightarrow \|\nabla F(u)\| \geq \nu_a |F(u) - F(a)|^{1-\theta}. \tag{48}$$

On the other hand, by using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} [\Phi_{\overline{\varepsilon}}(u,v) - \Phi_{\overline{\varepsilon}}(a,0)]^{1-\theta} &= \left[\frac{1}{2}\|v\|^2 + F(u) - F(a) + \varepsilon \|\nabla F(u)\|^\alpha \langle \nabla F(u), v\rangle\right]^{1-\theta} \\ &\leq \|v\|^{2(1-\theta)} + |F(u) - F(a)|^{1-\theta} + \|\nabla F(u)\|^{(\alpha+1)(1-\theta)}\|v\|^{1-\theta}. \end{aligned} \tag{49}$$

Thanks to Young's inequality we obtain

$$\|\nabla F(u)\|^{(\alpha+1)(1-\theta)}\|v\|^{1-\theta} \leq \|\nabla F(u)\| + \|v\|^{\frac{1-\theta}{\theta - \alpha(1-\theta)}}.$$

Then (49) becomes

$$[\Phi_{\overline{\varepsilon}}(u,v) - \Phi_{\overline{\varepsilon}}(a,0)]^{1-\theta} \leq \|v\|^{2(1-\theta)} + |F(u) - F(a)|^{1-\theta} + \|\nabla F(u)\| + \|v\|^{\frac{1-\theta}{\theta - \alpha(1-\theta)}}.$$

Since $2(1-\theta)$ and $\frac{1-\theta}{\theta - \alpha(1-\theta)}$ are bigger then 1, using also (48), we get for all $(u,v) \in B$ with $\|v\| \leq 1$ and $\|u - a\| < \delta_a$

$$\begin{aligned} [\Phi_{\overline{\varepsilon}}(u,v) - \Phi_{\overline{\varepsilon}}(a,0)]^{1-\theta} &\leq \|v\| + |F(u) - F(a)|^{1-\theta} + \|\nabla F(u)\| + \|v\| \\ &\leq \left(2 + \frac{1}{\nu_a}\right) [\|\nabla F(u)\| + \|v\|] \\ &\leq \frac{1}{\rho}\left(2 + \frac{1}{\nu_a}\right) \|\nabla \Phi_{\overline{\varepsilon}}(u,v)\| \quad \text{by (47).} \end{aligned}$$

Therefore $\Phi_{\overline{\varepsilon}}$ satisfies (10). Moreover since $\alpha < \frac{\theta}{1-\theta}$, we have

$$\beta(1-\theta) = (\alpha+1)(1-\theta) < \left(\frac{\theta}{1-\theta} + 1\right)(1-\theta) = 1,$$

and (11) is satisfied. All the assumptions of theorem 2.1 are thus satisfied

In order to get the speed of convergence given in the theorem 4.7, it suffices to remark that when $\beta = 1 + \alpha$ we have

$$\frac{1 - \beta(1-\theta)}{\beta(1-\theta) - \theta} = \frac{\theta - (1-\theta)\alpha}{1 - 2\theta + \alpha(1-\theta)}.$$

□

5. Numerical simulations.

5.1. **The finite dimensional case.** In this section we present some numerical results on the implicit scheme given by (25). The simulations were performed with C++ and python3. We dealt with more general situations of C^2 functions F than convex and coercive ones as well as we also performed some numerical simulations with a semi-implicit scheme (see below).

In the situation of (25), obtaining (u_{n+1}, v_{n+1}) from (u_n, v_n) has been done by means of a Newton method.

We have considered different F and different value of α, even with $\alpha > 1$. The Fs that we consider do not necessarily satisfy the conditions imposed in order to get convergences, but in many situations we observe numerical results that are conform to the predicted behavior.

Here are some pictures of the results of the simulations. In each, we have fixed a maximum number of iterations and a stopping condition on to the value of F on the sequences constructed with respect to the minimum value of F.

5.1.1. *Example 1.* We considered the following situation : See figures (1) and (2) and (3) and (4).

$$F : \mathbb{R}^2 \to \mathbb{R}$$

$$(u,v) \mapsto F(u,v) = \begin{cases} \frac{1}{2}(u^2+2v^2-1)^2 \text{ if } u^2+2v^2-1 \geq 0 \\ 0 \text{ elsewhere.} \end{cases} \tag{50}$$

$\alpha = 0.4$, $\Delta t = 0.01$ or 0.1, $N = 1000$ is the maximum number of iterations.

Figure 1. Simulation for (50). Here $\alpha = 0.4$ and $\Delta t = 0.01$, the dotted curve is the apparently decreasing relative energy $E(u_n, v_n)/E(u_0, v_0)$. The solid curve is the boundary of the zero level of F (which is in its interior). Convergence occurs to a critical point of F which may be in the interior of the zero level set of F.

odeex1alpha0dot41-eps-converted-to.pdf

Figure 2. $\alpha = 0.4$ and $\Delta t = 0.01$, the dotted curve is the value of the velocity v_n. The solid curve describes the coordinates of u_n. The dash-dotted curve corresponds to the sequence $(u_n, F(u_n, v_n))$.

odeex1alpha0dot42-eps-converted-to.pdf

FIGURE 3. Simulation for (50). Here $\alpha = 0.4$ and $\Delta t = 0.1$, the dotted curve is the relative energy $E(u_n, v_n)/E(u_0, v_0)$. The solid curve is the boundary of the zero level of F (which is in its interior). Convergence occurs to a critical point of F which seems to be on the boundary of the zero level of F.

odeex1alpha0dot43-eps-converted-to.pdf

FIGURE 4. Simulation for (50). Here $\alpha = 0.4$ and $\Delta t = 0.1$, the dotted curve is the value of the velocity v_n. The solid curve describes the coordinates of u_n. The dash-dotted curve corresponds to the sequence $(u_n, F(u_n, v_n))$.

odeex1alpha0dot44-eps-converted-to.pdf

5.1.2. *Example 2.* We also considered the following situation : See figures (5) and (6) and (7) and (8).

$$
\begin{aligned}
&F : \mathbb{R}^2 \to \mathbb{R} \\
&(u,v) \mapsto F(u,v) = \begin{cases} \frac{1}{2}((u-1)^2 + (v-1)^2 - 1)^2 \text{ if } (u-1)^2 + (v-1)^2 - 1 \geq 0 \\ 0 \text{ elsewhere.} \end{cases} \\
&\alpha = 0.4 \\
&\Delta t = 0.01 \text{ or } 0.1 \\
&N = 1000 \text{ is the maximum number of iterations.}
\end{aligned}
\tag{51}
$$

FIGURE 5. Simulation for (51). Here $\alpha = 0.4$ and $\Delta t = 0.01$, the dotted curve is the relative energy $E(u_n, v_n)/E(u_0, v_0)$ it is apparently decreasing. The solid curve is the boundary of the zero level of F (which is in its interior). Convergence occurs to a critical point of F which may be in the interior the zero level of F.

odeex2alpha0dot41-eps-converted-to.pdf

FIGURE 6. $\alpha = 0.4$ and $\Delta t = 0.01$, the dotted curve is the value of the velocity v_n. The solid curve describes the coordinates of u_n. The dash-dotted curve corresponds to the sequence $(u_n, F(u_n, v_n))$.

odeex2alpha0dot42-eps-converted-to.pdf

FIGURE 7. Simulation for (51). Here $\alpha = 0.4$ and $\Delta t = 0.1$, the dotted curve is the relative energy $E(u_n, v_n)/E(u_0, v_0)$. The solid curve is the boundary of the zero level of F (which is in its interior). Convergence occurs to a critical point of F which seems to be on the boundary of the zero level of F.

odeex2alpha0dot43-eps-converted-to.pdf

FIGURE 8. Simulation for (51). Here $\alpha = 0.4$ and $\Delta t = 0.1$, the dotted curve is the value of the velocity v_n. The solid curve describes the coordinates of u_n. The dash-dotted curve corresponds to the sequence $(u_n, F(u_n, v_n))$.

odeex2alpha0dot44-eps-converted-to.pdf

5.1.3. *Example 3.* We also dealt with the following situation where F is C^2 and non-convex : See figure (9)

$$
\begin{aligned}
&F : \mathbb{R}^2 \to \mathbb{R} \\
&(u, v) \mapsto F(u, v) = \frac{1}{2}((u-1)^2 + (v-1)^2 - 1)^2 \\
&\alpha = 0.4 \\
&h = 0.01 \\
&N = 1000 \text{ is the maximum number of iterations.}
\end{aligned} \tag{52}
$$

We also simulated the following : See figure (10)

FIGURE 9. $\alpha = 0.4$. We again observe the numerical convergence, but have no proof yet.

odeex2alpha0dot4deltat0dot01-eps-converted-to.pdf

$$
\begin{aligned}
F &: \mathbb{R}^2 \to \mathbb{R} \\
(u,v) &\mapsto F(u,v) = ((u-1)^2 + (v-1)^2 - 1)^2 \\
\alpha &= 0.4 \\
h &= 0.1 \\
N &= 1000 \text{ is the maximum number of iterations.}
\end{aligned} \tag{53}
$$

Here again we observe the numerical convergence.

Figure 10. $\alpha = 0.4$

odeex2alpha0dot4deltat0dot1-eps-converted-to.pdf

5.2. **Semi-implicit scheme.** Let us also mention that we performed some simulations for the semi-implicit case, that is

$$
\left\{
\begin{aligned}
\frac{u_{n+1} - u_n}{\Delta t} &= v_{n+1} \\
\frac{v_{n+1} - v_n}{\Delta t} &= -\|v_{n+1}\|^{\alpha} v_{n+1} - \nabla F(u_n) \\
u_0,\ v_0 &\in \mathbb{R}^d
\end{aligned}
\right. \tag{54}
$$

Though we do not have proof of the convergence of sequences satisfying (54) with assumptions similar to the ones for the implicit scheme, we believe that convergence holds as the following examples show.

5.2.1. *Example 1.* One of these example is the following : See figure (11)

$$
\begin{aligned}
F &: \mathbb{R}^2 \to \mathbb{R} \\
(u,v) &\mapsto F(u,v) = ((u-1)^2 + (v-1)^2 - 1)^2 \\
\alpha &= 0.5 \\
h &= 0.01 \\
N &= 1000 \text{ is the number of iterations.}
\end{aligned} \tag{55}
$$

Figure 11. $\alpha = 0.5$

Ex1-2d.jpg

5.2.2. *Example 2.* See figure (12)

$$\begin{aligned} F &: \mathbb{R}^2 \to \mathbb{R} \\ (u,v) &\mapsto F(u,v) = ((u-1)^2 + (v-1)^2 - 1)^2 \\ \alpha &= 1.5 \\ h &= 0.001 \\ N &= 10000 \text{ is the number of iterations.} \end{aligned} \tag{56}$$

Figure 12. $\alpha = 1.5$

Ex2-2d.jpg

5.2.3. *Example 3.* See figure (13)

$$\begin{aligned} F &: \mathbb{R}^2 \to \mathbb{R} \\ (u,v) &\mapsto F(u,v) = \frac{u^4}{4} + \frac{u^2}{4} - \frac{u^2 v}{2} \\ \alpha &= 0.5 \\ h &= 0.001 \\ N &= 10000 \text{ is the number of iterations.} \end{aligned} \tag{57}$$

Figure 13. $\alpha = 1.5$

Ex3-2d.jpg

5.3. **Approximating the nonlinear nonlocal wave equation.** We consider here the nonlinear wave equation set on a regular bounded connected subset of $\mathbb{R}^N$. Moreover the nonlinearity that we consider here is nonlocal.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \|\dfrac{\partial u}{\partial t}\|_2^\alpha \dfrac{\partial u}{\partial t} - \Delta u + f'(u) = 0, \; (t,x) \in (0,\infty) \times \Omega, \\ u(t,x) = 0, \; t > 0, \; x \in \partial\Omega, \\ u(0,x) = u^0(x), \; \dfrac{\partial u}{\partial t}(0,x) = u^1(x). \end{cases} \tag{58}$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ a C^2.

The reason for which the nonlinearity involves a nonlocal term (*i.e.* a L^2-norm) comes from the fact the discretization that we are going to consider leads to a similar term as in

(25).
Let us recall that for the equation (58) some convergence results were obtained in [3].
We wish to approximate the equation (58) by means of an implicit difference scheme in time and a finite-element method in space.
For this we classically rewrite (58) as a system

$$\begin{cases} \dfrac{\partial v}{\partial t} + \|v\|_2^\alpha v - \Delta u + f'(u) = 0, \ (t,x) \in (0,\infty)\times\Omega, \\ \dfrac{\partial v}{\partial t} = u, \ (t,x) \in (0,\infty)\times\Omega, \\ u(t,x) = 0, \ t > 0, \ x \in \partial\Omega, \\ u(0,x) = u_0(x), \ \dfrac{\partial u}{\partial t}(0,x) = u_1(x). \end{cases} \tag{59}$$

We will restrict ourselves to the situation when $u_i \in H_0^1(\Omega)$ for $i = 0, 1$.
The semi-implicit finite difference scheme in time that we consider consists of, given some time step h, a sequence (u_n, v_n) of elements of $H_0^1(\Omega)^2$ such that

$$\begin{cases} \dfrac{v_{n+1} - v_n}{h} + \|v_{n+1}\|_2^\alpha v_{n+1} - \Delta u_{n+1} + f'(u_{n+1}) = 0 \\ \dfrac{u_{n+1} - u_n}{h} = v_{n+1} \\ u_0 = u^0 \\ u_1 = u^1. \end{cases} \tag{60}$$

We consider now V a finite dimensional subspace of $H_0^1(\Omega)$ of dimension N and take a basis of V, $(\phi_k)_{k=1,\dots,N}$, orthonormal in $L^2(\Omega)$.
We will approximate the implicit difference scheme in time by considering the following finite-element approximation of (60) We will now consider $(U_n, V_n) \in V \times V$ such that $\forall i = 1, ..., N$

$$\begin{cases} \displaystyle\int_\Omega \frac{V_{n+1} - V_n}{h}\phi_i + \int_\Omega \|V_{n+1}\|_2^\alpha V_{n+1}\phi_i + \int_\Omega \left(\nabla U_{n+1}\nabla\phi_i + f'(U_{n+1})\phi_i\right) dx = 0 \\ \displaystyle\int_\Omega \frac{U_{n+1} - U_n}{h}\phi_i dx = \int_\Omega V_{n+1}\phi_i dx \end{cases} \tag{61}$$

If we write $V_n = \displaystyle\sum_{k=1}^N V_{n,k}\phi_k$, and $U_n = \displaystyle\sum_{k=1}^N U_{n,k}\phi_k$ then (61) becomes

$$\begin{cases} \dfrac{V_{n+1,k} - V_{n,k}}{h} + \|V_{n+1}\|^\alpha V_{n,k} + (\nabla\mathcal{F}(V_{n+1}))_k = 0 \\ \dfrac{U_{n+1,k} - U_{n,k}}{h} = V_{n+1,k} \\ k = 1, ..., N \end{cases} \tag{62}$$

where

$$\mathcal{F} : V \to \mathbb{R}$$
$$v \mapsto \mathcal{F}(v) = \frac{1}{2}\int_\Omega (\|\nabla v\|^2 + f(v))dx.$$

It can be easily seen that if f satisfies the conditions (26) and (27) so does $\mathcal{F}$.

In this form, we see that (62) stands as the system (25) and thus provided f satisfies the assumptions required for the theorem 4.7, the results of convergence apply. Let us thus remark that in order to guarantee that $\mathcal{F}$ satisfies (31) it is sufficient to assume that f is real analytic. This assumption is also made for the related situation in [9].

We have simulated the wave equation in dimension 1 with different examples.

Unfortunately some of the simulations performed (for some specific initial data which we will not present here) lacks to have the asymptotic behavior that is expected by the theoretical result, on the contrary to what happens when there are no nonlocal terms. We were not able though to overcome, without artifacts, these oscillations on standard computers. In order to get some numerical asymptotic convergence, in many cases adding some regularization on the velocity seems to be sufficient. For example, if one denotes (v_n^i) the computed velocity at each discrete point of $[0,1]$, at time ih, one may consider an intermediate step and define new values $(v_n^{i+1/2})$ as means of the (v_n^i), *e.g.* $v_n^{i+1/2} = \alpha_1 v_{n-1}^i + \alpha_2 v_n^i + \alpha_3 v_{n+1}^i$ where $\alpha_j, j = 1,2,3$ are nonnegative numbers such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$. The asymptotic behavior of the corresponding continuous equation has not, to our knowledge been studied. In that situation the continuous equation would be related to

$$\frac{\partial^2 u}{\partial t^2} + \|\frac{\partial u}{\partial t}\|_2^\alpha A(\frac{\partial u}{\partial t}) - \Delta u + f'(u) = 0$$

where A is defined by $A(h)$ the solution of $-\Delta A(h) = h$ with either 0 boundary condition on a cell or the whole domain of study.

We present some very rough numerical results without proceeding to the aforementioned regularization.

Though we performed different simulations with different initial data and velocity, with $\alpha = 0.5$, typically u_0 and u_1 are of the form

$$x \longmapsto \lambda x(x-1)(x-1/2)$$

or some regular approximations of

$$x \longmapsto \lambda \min(x, 1-x)$$

or

$$x \longmapsto \lambda(x(1-x) + \sin(\pi x))$$

with different λs for u_0 and u_1, we show here a few among them.

We have considered $f(x) = x - \sin(x)$ though it does not satisfy our assumptions on f.

Here are some results of simulations :

In the case of figures (14) up to (18), we also computed the L^2 norm of the function determined by U_n which seems to be decreasing after some iterations in time. We take Δt of the order of $1/M^2$ where M is the number of points of discretization in $[0,1]$.

FIGURE 14. $\alpha = 0.5$, $T = 9/16$, $f(x) = x - \sin(x)$.

edononmoy1-900-eps-converted-to.pdf

FIGURE 15. $\alpha = 0.5$, $T = 9/8$, $f(x) = x - \sin(x)$.

edononmoy1-1800-eps-converted-to.pdf

FIGURE 16. $\alpha = 0.5$, $T = 45/8$, $f(x) = \sin(x) - x$. Here we can see some oscillations appearing, though there seem to have convergence to 0.

edononmoy1-9000-eps-converted-to.pdf

FIGURE 17. $\alpha = 0.5$, $T = 9/16$, $f(x) = \sin(x) - x$.

edononmoy2-900-eps-converted-to.pdf

FIGURE 18. $\alpha = 0.5$, $T = 45/8$, $f(x) = \sin(x) - x$.

edononmoy2-9000-eps-converted-to.pdf

6. Appendix

Proof of the lemma 4.4. We pose $h = v - u$. By using the Taylor formula, we get

$$F(v) = F(u+h) = F(u) + \int_0^1 \langle \nabla F(u+sh), h \rangle \, ds.$$

Then we have

$$\begin{aligned} F(v)-F(u)-\langle\nabla F(u),v-u\rangle &= \int_0^1 \langle\nabla F(u+sh)-\nabla F(u),h\rangle\,ds \\ &\geq \int_0^1 -c_F s^{\alpha+1}\|h\|^{\alpha+2}\,ds \quad \text{by (26)} \\ &\geq -\frac{c_F}{\alpha+2}\|u-v\|^{\alpha+2}. \end{aligned}$$

□

Proof of the proposition 4.3. The first part of the proposition is a result of D'Acunto and Kurdyka (see [7]). The proof of the second part can be found in [5], we give it for completeness.

Since F is analytic, then by using the result of Lojasiewicz [15, 16], we have for all $a \in S$

$$\exists\theta_a \in (0,\frac{1}{2}]\ \exists\delta_a > 0/\ \forall u \in B(a,\delta_a) \quad \|\nabla F(u)\| \geq |F(u)-F(a)|^{1-\theta_a}. \tag{63}$$

As $S \subset \bigcup_{x\in S} B(x,\delta_x)$ and S is compact, then there $x_1,\cdots,x_p \in S$ such that

$$S \subset \bigcup_{j=1}^{p} B(x_j,\delta_{x_j}).$$

Let $a \in S$, then there exists $j \in \{1,\cdots,p\}$ such that $a \in B(x_j,\delta_{x_j})$. Using (63) we deduce that $F(a) = F(x_j)$. On the other hand, since F is continuous, then there exists $\mu > 0$ such that

$$\forall u \in B(a,\mu) \quad |F(u)-F(a)| \leq 1. \tag{64}$$

Let $\sigma_a = \inf\big(\mu, \delta_{x_j} - \|a-x_j\|\big)$. Then obviously we have $B(a,\sigma_a) \subset B(x_j,\delta_{x_j}) \cap B(a,\mu)$. Using once again (63), we have for all $u \in B(a,\sigma_a)$

$$\begin{aligned} |F(u)-F(a)| &\leq |F(u)-F(x_j)| + |F(x_j)-F(a)| \\ &\leq \|\nabla F(u)\|^{\frac{1}{1-\theta_{x_j}}} \end{aligned}$$

Let $\theta = \min_{j\in\{1,\cdots,p\}} \theta_{x_j}$. By (64) we conclude that for all $\in B(a,\sigma_a)$

$$|F(u)-F(a)|^{1-\theta} \leq |F(u)-F(a)|^{1-\theta_{x_j}} \leq \|\nabla F(u)\|.$$

□

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