H_∞ interval observer design for uncertain discrete-time linear switched systems with unknown inputs

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***Abstract*—This paper deals with unknown input interval observers for discrete-time linear switched systems. A new structure of interval observer is used to overcome the design difficulty of coordinate transformation often used in such context. The interval observer gains are computed by solving Linear Matrix Inequalities (LMI) derived from multiple quadratic Lyapunov functions (MQLF) under average dwell time switching signals.**

***Index Terms*—component, formatting, style, styling, insert**

I. Introduction

Interval observers present a potentiel solution to deal with systems affected by various types of disturbances and measurement noise. They rely on the design of a dynamic structure with two outputs giving an upper and a lower of the actual state [16]. Note that this cutting-edge class of observers requires the knowledge of bounds of the initial state values as well as bounds of measurement noise and disturbances. In return for these requirements which are generally satisfied in real-life applications, interval estimation addresses weak points of classical observers which give only asymptotic estimates in the absence of disturbances. That is why the technique originating in [9] has been developed in many directions, e.g., a family of linear systems [2], [6], [24], some classes of nonliear systems [17], fuzzy systems [18] and other conerns such as monitoring, fault detection and control purposes [4], [5] etc.

Many of the dynamical systems encountered in practice are of hybrid nature. Recently, framers and interval observers design for switched systems which consist of a finite number of subsystems governed by switching signals have received a great interest [7], [8], [14], [15], [19]. Usually, it is not very difficult to achieve the framer property, which is the notion of providing intervals in which state variables stay, if one does not care about the length of estimated intervals (i.e., the stability property). For switched systems, the most challenging step in interval observer designs is to ensure this last property. Most of the above works are based on the positivity of the estimation errors after a coordinate transformation which may cause conservatisms: indeed, it is often hard to design simultaneously observer gains and changes of coordinates ensuring at the same time the positivity property and good estimation accuracy.

Note that in addition to noise and disturbances, real systems are often subject to unknown inputs. Such a case has been already investigated for non switched systems (the reader can for instance refer to [3], [12], [13], [23]). Furthermore, some works attempted to consider the case of continuous-time switched systems with unknown inputs [21], [22]. However, to the best of the authors' knowledge, the case of interval observers design for discrete-time switched systems subject to unknown inputs have not yet been fully considered in the literature. It is worth pointing out that the estimators proposed in the present work are not derived directly because changing the system from continuous to discrete time not only raises changes of stability properties but also requires the estimation procedure of the unknown input to be properly adjusted.

Moreover, inspired by [20], a new structure providing more design degrees of freedom than existing works in the literature is introduced. This study has three main contributions: first, a new observer structure, that not only provides more design degrees of freedom but also relaxes the design conditions, is developed. Second, the construction simultaneously returns interval estimates of states and unknown inputs for a class of uncertain discrete-time linear switched systems. Finally, the method takes into account the effects of process disturbances and measurement noise by incorporating H_∞ technique to attenuate uncertainties in order to obtain accurate interval estimation.

The remainder of this paper is organized as follows. Some preliminaries are introduced in Section II. Section III introduces the structure of an interval observer allowing the estimation of the state and the reconstruction of the unknown input based on the use of H_∞ formalism. Section IV draws simulation results to illustrate the different steps of the proposed design. Section V gives the conclusion and perspectives.

II. Preliminaries

A. Notation, definitions, basic result

The set of natural numbers, integers and real numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$, respectively. The set of nonnegative real numbers and nonnegative integers are denoted by $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$, respectively. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $|x|$, and for a measurable and locally essentially bounded input $u : \mathbb{Z} \to \mathbb{R}$, the symbol $\|u\|_{[t_0,t_1]}$ denotes its L_∞ norm, $\|u\|_{[t_0,t_1]} = \sup\{|u|, t \in [t_0, t_1]\}$. If $t_1 = \infty$ then we will simply write $\|u\|$. We denote $\mathcal{L}_\infty$ as the set of all inputs u with the property $\|u\| < \infty$. We denote the sequence of integers $1, \ldots, N$ as $\overline{1, N}$. Inequalities must be understood *component-wise*, i.e., for $x_a = [x_{a,1}, ..., x_{a,n}]^\top \in \mathbb{R}^n$ and $x_b = [x_{b,1}, ..., x_{b,n}]^\top \in \mathbb{R}^n$, $x_a \leq x_b$ if and only if, for all $i \in \overline{1, N}$, $x_{a,i} \leq x_{b,i}$. For a square matrix $Q \in \mathbb{R}^{n\times n}$, let the matrix $Q^+ \in \mathbb{R}^{n\times n}$ denote $Q^+ = (\max\{q_{i,j}, 0\})_{i,j=1,1}^{n,n}$, where the notation $Q = (q_{i,j})_{i,j=1,1}^{n,n}$ is used. Let $Q^- \in \mathbb{R}^{n\times n}$ be defined by $Q^- = Q^+ - Q$ and the matrix of absolute values of all elements be defined by $|Q| = Q^+ + Q^-$, the superscripts $+$ and $-$ for other purposes are defined appropriately when they appear. The asterisk $\star$ denotes the symmetric term in a symmetric matrix. A square matrix $Q \in \mathbb{R}^{n\times n}$ is said to be nonnegative if all its entries are nonnegative. 0 and I are respectively the zero and identity matrix of appropriate dimensions. A positive (res. negative) (semi) definte matrix $P \in \mathbb{R}^{n\times n}$ is denoted as $P \succ (\succcurlyeq)\ 0$ (resp. $P \prec (\preccurlyeq)\ 0$). For a non-square matrix B, its left pseudo-inverse is $B^\dagger = (B^T B)^{-1} B^T$.

Lemma 1. *[2] Consider a vector $x \in \mathbb{R}^n$ such that $\underline{x} \leq x \leq \overline{x}$ and a constant matrix $A \in \mathbb{R}^{n\times n}$, then*

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}, \tag{1}$$

with $A^+ = \max\{0, A\}$, $A^- = A^+ - A$.

Lemma 2. *[16] A system described by $x(k+1) = Ax(k) + u(k)$, with $x(k) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, is nonnegative if and only if the matrix A is elementwise nonnegative, $u(k) \geq 0$ and $x(k_0) \geq 0$. In this case, the system is also called cooperative.*

Lemma 3. *[1] Given matrices $\Xi \in \mathbb{R}^{a\times b}$, $\Psi \in \mathbb{R}^{b\times c}$ and $\Upsilon \in \mathbb{R}^{a\times c}$ with $rank(\Psi) = c$. The general solution Ξ of the equation $\Xi\Psi = \Upsilon$ is*

$$\Xi = \Upsilon\Psi^\dagger + S\left(I - \Psi\Psi^\dagger\right) \tag{2}$$

where $S \in \mathbb{R}^{a\times b}$ is an arbitrary matrix.

B. Average dwell time

Definition 1. *[10] For a switching signal σ and any $0 \leq k_l \leq k_s$, let $N_\sigma(k_l, k_s)$ denote the number of discontinuities of σ on the interval $[k_l, k_s)$. If there exist a scalar $\tau_a > 0$ and an integer $N_0 \geq 0$, such that*

$$N_\sigma(k_l, k_s) \leq N_0 + \frac{k_s - k_l}{\tau_a} \tag{3}$$

holds for all k_l and k_s, then the scalar $\tau_a > 0$ is called an average dwell time (ADT) and N_0 the chatter bound. In this paper, we assume that $N_0 = 0$ for simplicity as commonly used in the literature.

C. Input to state stability

Lemma 4. *[11] Consider the discrete-time switched system $x(k+1) = f_{\sigma(k)}(\xi(k), \eta(k))$, $\sigma(k) \in \overline{1, N}$. Suppose that there exists $\mathcal{C}^1$ functions $V_{\sigma(k)} : \mathbb{R}^n \longrightarrow \mathbb{R}_+$, class $\mathcal{K}_\infty$ functions α_1, α_2, γ and constants $0 < \alpha < 1$, $\mu \geq 1$ such that $\forall \xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^l$ we have*

$$\alpha_1(\|\xi\|) \leq V_{\sigma(k)}(\xi) \leq \alpha_2(\|\xi\|), \tag{4}$$

$$V_{\sigma(k)}(\xi(k+1)) - V_{\sigma(k)}(\xi(k)) \leq -\alpha V_{\sigma(k)}(\xi(k)) + \varrho(\|\eta\|), \tag{5}$$

and for each switching instant k_l, $l = 0, 1\ 2, 3, \ldots$,

$$V_{\sigma(k_l)}(\xi(k)) \leq \mu V_{\sigma(k_l - 1)}(\xi(k)). \tag{6}$$

Then the system $x(k+1) = f_{\sigma(k)}(\xi(k), \eta(k))$, $\sigma(k) \in \overline{1, N}$ is Input-to-State Stable (ISS) for any switching signal satisfying the average dwell time

$$\tau_a \geq \tau_a^* = -\frac{\ln(\mu)}{\ln(1-\alpha)}. \tag{7}$$

III. Main results

Consider the following discrete-time linear switched system

$$\begin{cases} x(k+1) = A_{\sigma_k}x(k) + B_{\sigma_k}u(k) + D_{\sigma_k}d(k) + \omega(k), \\ y(k) = C_{\sigma_k}x(k) + v(k), \sigma_k \in \overline{1, N},\ N \in \mathbb{N}. \end{cases} \tag{8}$$

with $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, $\omega \in \mathbb{R}^n$ and $v \in \mathbb{R}^p$ are respectively the disturbances and the measurement noise. $d \in \mathbb{R}^l$ is the unknown input. $\sigma_k = \sigma(k)$ is a known piecewise constant function that takes its values in an index set $\overline{1, N}$, $N > 1$, where σ_k is the index of the active subsystem and N is the number of subsystems. A_{σ_k}, B_{σ_k} and C_{σ_k} and D_{σ_k} are time-invariant matrices of the corresponding dimensions. The σ_k^{th} is said to be active when $\sigma_k \in \overline{1, N}$ and $k \in [k_{\sigma_k},\ k_{\sigma_{k+1}})$. Some assumptions are introduced.

Assumption 1. *The switching signal $\sigma(k)$ is assumed to be known.*

Assumption 2. *The initial condition, the disturbances and the measurement noise are assumed to be bounded such that*

$$\underline{x}_0 \leq x(0) \leq \overline{x}_0,\ \forall k \geq 0, \tag{9}$$

$$-\overline{\omega} \leq \omega(k) \leq \overline{\omega},\ \forall k \geq 0, \tag{10}$$

$$-\overline{v} \leq v(k) \leq \overline{v},\ \forall k \geq 0, \tag{11}$$

where $\underline{x}_0$, $\overline{x}_0$, $\overline{\omega} \in \mathbb{R}^n$ and $\overline{v} \in \mathbb{R}^p$ are known vectors.

Assumption 3. $\forall\ \sigma_k \in \overline{1,N},\ N \in \mathbb{N}$, *the matrices C_{σ_k} and D_{σ_k}, are full row rank and full column rank respectively.*

Interval observers overcome weak points of classical observers. They can cope with large disturbances and give componentwise information on the range of the possible solutions at any time instant at the cost of restrictive assumptions. Indeed, the interval property requires error systems to be positive. Although some relaxing techniques are available, securing the positivity at some point during the design process remains the key. Motivated by [20], this paper looks for better tricks in achieving the positivity without employing changes of coordinates frequently employed in the literature. We introduce new matrices to relax design conditions after decoupling the unknown input from the studied system (8) by considering it as an auxiliary state. In fact, (8) can be rewritten as

$$\begin{cases} E_{\sigma_k}\tilde{x}(k+1) & = \tilde{A}_{\sigma_k}\tilde{x}(k) + \tilde{B}_{\sigma_k}u(k) + \tilde{I}\omega(k), \\ y(k) & = \tilde{C}_{\sigma_k}\tilde{x}(k) + v(k), \end{cases} \tag{12}$$

where

$$\tilde{x}(k+1) = \begin{bmatrix} x(k+1) \\ d(k) \end{bmatrix}, \tilde{x}(0) = \begin{bmatrix} x(0) \\ 0 \end{bmatrix},$$
$$E_{\sigma_k} = \begin{bmatrix} I & -D_{\sigma_k} \\ 0 & 0 \end{bmatrix}, \tilde{I} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$
$$\tilde{A}_{\sigma_k} = \begin{bmatrix} A_{\sigma_k} & 0 \\ 0 & 0 \end{bmatrix}, \tilde{B}_{\sigma_k} = \begin{bmatrix} B_{\sigma_k} \\ 0 \end{bmatrix}, \tilde{C}_{\sigma_k} = \begin{bmatrix} C_{\sigma_k} & 0 \end{bmatrix}.$$

By designing the interval observer of the augmented state $\tilde{x}(k+1)$, i.e., two bounds $\underline{\tilde{x}}(k)$, $\overline{\tilde{x}}(k)$ such that

$$\underline{\tilde{x}}(k) \leq \tilde{x}(k) \leq \overline{\tilde{x}}(k), \forall k \in \mathbb{Z}_+, \tag{13}$$

one can immediately deduce the bounds that enclose the state and the unknown input.

Remark 1. *By augmenting unknown input $d(k)$ as a part of the state vector $\tilde{x}(k+1)$, the structural conditions for decoupling unknown input in [3], [12] are relaxed. Subsequently, the proposed method possesses a wider application scope than the above-mentioned works.*

In the sequel, a new framer candidate is introduced for the augmented state (12) and a sufficient condition is given such that the framer becomes an interval observer.

A. Framer design

As a solution to (12), the following framer candidate is considered

$$\begin{cases} \overline{\xi}(k+1) & = T_{\sigma_k}\tilde{A}_{\sigma_k}\overline{\tilde{x}}(k) + T_{\sigma_k}\tilde{B}_{\sigma_k}u(k) \\ & \quad + L_{\sigma_k}(y(k) - \tilde{C}_{\sigma_k}\overline{\tilde{x}}(k)) + \Delta \\ \overline{\tilde{x}}(k) & = \overline{\xi}(k) + N_{\sigma_k}y(k) \\ \underline{\xi}(k+1) & = T_{\sigma_k}\tilde{A}_{\sigma_k}\underline{\tilde{x}}(k) + T_{\sigma_k}\tilde{B}_{\sigma_k}u(k) \\ & \quad + L_{\sigma_k}(y(k) - \tilde{C}_{\sigma_k}\underline{\tilde{x}}(k)) - \Delta \\ \underline{\tilde{x}}(k) & = \underline{\xi}(k) + N_{\sigma_k}y(k) \end{cases} \tag{14}$$

with

$$\Delta = |T_{\sigma_k}\tilde{I}|\overline{\omega} + |L_{\sigma_k}|\overline{v} + |N_{\sigma_k}|\overline{v} \tag{15}$$

where L_{σ_k} is an appropriate observer gain associated to the σ_k-subsystem with $\sigma_k \in \overline{1,N}$ to be computed later.

The matrices T_{σ_k}, N_{σ_k}, with $\sigma_k \in \overline{1,N}$, are computed satisfying the following condition

$$T_{\sigma_k}E_{\sigma_k} + N_{\sigma_k}\tilde{C}_{\sigma_{k+1}} \quad = I \tag{16}$$

Theorem 1. *Let Assumptions 1-3 hold, the lower bound $\underline{\tilde{x}}(k)$ and upper bound $\overline{\tilde{x}}(k)$ for the state $\tilde{x}(k)$ given by (14) satisfy (13), if* (16) *hold and* $(T_{\sigma_k}\tilde{A}_{\sigma_k} - L_{\sigma_k}\tilde{C}_{\sigma_k}) \geq 0$, $\forall\ \sigma_k \in \overline{1,N}$ *provided that* $\underline{\tilde{x}}_0 := \begin{bmatrix} \underline{x}(0) \\ 0 \end{bmatrix} \leq \tilde{x}(0) \leq \overline{\tilde{x}}_0 := \begin{bmatrix} \overline{x}(0) \\ 0 \end{bmatrix}$.

Proof. Let $\overline{e}(k) = \overline{\tilde{x}}(k) - \tilde{x}(k)$ and $\underline{e}(k) = \tilde{x}(k) - \underline{\tilde{x}}(k)$ be the upper observation and the lower observation errors, respectively. The aim is to prove that $\overline{e}(k)$ and $\underline{e}(k)$ are nonnegative. Bearing in mind (16), the dynamic of the upper error follows

$$\begin{aligned} \overline{e}(k+1) \quad & = (T_{\sigma_k}\tilde{A}_{\sigma_k} - L_{\sigma_k}\tilde{C}_{\sigma_k})\overline{e}(k) + \Delta \\ & -T_{\sigma_k}\tilde{I}\omega(k) + L_{\sigma_k}v(k) + N_{\sigma_k}v(k+1). \end{aligned} \tag{17}$$

Similarly, the dynamic of the lower error is described by

$$\begin{aligned} \underline{e}(k+1) \quad & = (T_{\sigma_k}\tilde{A}_{\sigma_k} - L_{\sigma_k}\tilde{C}_{\sigma_k})\underline{e}(k) + \Delta \\ & +T_{\sigma_k}\tilde{I}\omega(k) - L_{\sigma_k}v(k) - N_{\sigma_k}v(k+1) \end{aligned} \tag{18}$$

Taking in consideration *Lemma* 1, we have

$$\Delta - T_{\sigma_k}\tilde{I}\omega(k) + L_{\sigma_k}v(k) + N_{\sigma_k}v(k+1) \geq 0 \tag{19}$$

$$\Delta + T_{\sigma_k}\tilde{I}\omega(k) - L_{\sigma_k}v(k) - N_{\sigma_k}v(k+1) \geq 0 \tag{20}$$

From the fact that $\overline{e}(0) = \overline{\tilde{x}}(0) - \tilde{x}(0) \geq 0$ and $\underline{e}(0) = \tilde{x}(0) - \underline{\tilde{x}}(0) \geq 0$, it follows that, for all $k \in \mathbb{Z}_+$, $\overline{e}(k) \geq 0$ and $\underline{e}(k) \geq 0$. This ends the proof. □

Remark 2. *The main difference between the approach used in literature and the one presented in* (14) *is the introduction of additional parameters T_{σ_k}, N_{σ_k} in the framer structure. If we choose $T_{\sigma_k} = I$ and $N_{\sigma_k} = 0$ for all $\sigma_k \in \overline{1,N}$,* (13) *reduces to the interval observer presented in [15].*

B. Interval observer design using H_∞ performance

This part is devoted to the computation of gains L_{σ_k} using a Multiple Quadratic Lyapunov function such that the framer (13) becomes an interval observer. In order to optimize the width of the interval estimation, a $\gamma-$ performance is introduced.

Let us define the estimation error as follows

$$e(k)=\overline{e}(k)-\underline{e}(k) \tag{21}$$

Thus,

$$e(k+1)=(T_{\sigma_k}\tilde{A}_{\sigma_k}-L_{\sigma_k}\tilde{C}_{\sigma_k})e(k)+\Phi_{\sigma_k}\delta(k) \tag{22}$$

with

$$\delta(k)=\begin{bmatrix} -T_{\sigma_k}\tilde{I}\omega(k) \\ v(k) \\ v(k+1) \end{bmatrix} \tag{23}$$

and

$$\Phi_{\sigma_k}=2\begin{bmatrix} I & L_{\sigma_k} & N_{\sigma_k} \end{bmatrix} \tag{24}$$

Theorem 2 gives a formulation to select the gains L_{σ_k} for (14) such that $e(k)$ is ISS in the sense of Lemma 4.

Theorem 2. *Assume that all assumptions of Theorem 1 hold. For given scalars $\gamma>0$ and $0<\alpha<1$, if there exist positive scalars $\alpha_2>\alpha_1>0$, a diagonal matrix $P_{\sigma_k}\in\mathbb{R}^{n\times n}$ such that $P_{\sigma_k}\succ 0$, $W_{\sigma_k}\in\mathbb{R}^{n\times n}$, $G_{\sigma_k}\in\mathbb{R}^{n\times p}$ and $H_{\sigma_k}\in\mathbb{R}^{n\times(n+p)}$ such that*

$$P_{\sigma_k}\,\Theta^{\dagger}_{\sigma_k}\lambda_1\tilde{A}_{\sigma_k}+H_{\sigma_k}\psi_{\sigma_k}\lambda_1\tilde{A}_{\sigma_k}-G_{\sigma_k}\tilde{C}_{\sigma_k}\geq 0,\ \forall\sigma_k\in\overline{1,N} \tag{25}$$

$$\alpha_1 I\leq P_{\sigma_k}\leq\alpha_2 I,\ \forall\sigma_k\ \in\overline{1,N} \tag{26}$$

$$\begin{bmatrix} W_{\sigma_l} & P_{\sigma_k} \\ P_{\sigma_k} & P_{\sigma_k} \end{bmatrix}\succeq 0 \tag{27}$$

$$\begin{bmatrix} -(1-\alpha)P_{\sigma_k} & \star & \star & \star & \star \\ 0 & -\gamma^2 I & \star & \star & \star \\ 0 & 0 & -\gamma^2 I & \star & \star \\ 0 & 0 & 0 & -\gamma^2 I & \star \\ \kappa_{1\sigma_k} & 2P_{\sigma_k} & 2G_{\sigma_k} & 2\kappa_{2\sigma_k} & -P_{\sigma_k} \end{bmatrix}\preccurlyeq 0, \tag{28}$$

with

$$W_{\sigma_l}=\mu P_{\sigma_l},G_{\sigma_k}=P_{\sigma_k}L_{\sigma_k},\ H_{\sigma_k}=P_{\sigma_k}S_{\sigma_k},\ \forall\sigma_k,\ \sigma_l\in\overline{1,N}$$

$$\begin{aligned}\kappa_{1\sigma_k}&=P_{\sigma_k}\ \Theta^{\dagger}_{\sigma_k}\lambda_1\tilde{A}_{\sigma_k}+H_{\sigma_k}\psi_{\sigma_k}\lambda_1\tilde{A}_{\sigma_k}-G_{\sigma_k}\tilde{C}_{\sigma_k}\\ \kappa_{2\sigma_k}&=P_{\sigma_k}\ \Theta^{\dagger}_{\sigma_k}\lambda_2+H_{\sigma_k}\psi_{\sigma_k}\lambda_2,\ \forall\sigma_k\in\overline{1,N}\end{aligned}$$

and

$$\Theta_{\sigma_k}=\begin{bmatrix} E_{\sigma_k} \\ \tilde{C}_{\sigma_{k+1}} \end{bmatrix},\lambda_1=\begin{bmatrix} I \\ 0 \end{bmatrix},$$

$$\psi_{\sigma_k}=I-\Theta_{\sigma_k}\Theta^{\dagger}_{\sigma_k},\ \forall\sigma_k\in\overline{1,N}$$

Then, (14) *is an interval observer for* (8)*. Moreover, the optimal observer gain matrix*

$$L_{\sigma_k}=P^{-1}G_{\sigma_k},\ \forall\sigma_k\in\overline{1,N} \tag{29}$$

is computed via the solution of the following constrained minimization problem

$$\begin{aligned}&\underset{P_{\sigma_k},G_{\sigma_k},H_{\sigma_k}}{\textit{minimize}} && \beta\mu+(1-\beta)\gamma,\ \sigma_k=1,\dots,N\\ &\textit{subject to} && (25),\ (27),\ (28).\end{aligned} \tag{30}$$

Proof. Assumption (16) can be rewritten as

$$\begin{bmatrix} T_{\sigma_k} & N_{\sigma_k} \end{bmatrix}\begin{bmatrix} E_{\sigma_k} \\ \tilde{C}_{\sigma_{k+1}} \end{bmatrix}=I \tag{31}$$

Let Lemma 3 hold, then one can deduce that the matrices T_{σ_k}, N_{σ_k} are given as

$$\begin{aligned}\begin{bmatrix} T_{\sigma_k} & N_{\sigma_k} \end{bmatrix}&=I\begin{bmatrix} E_{\sigma_k} \\ \tilde{C}_{\sigma_{k+1}} \end{bmatrix}^{\dagger}\\ &+S_{\sigma_k}\left(I-\begin{bmatrix} E_{\sigma_k} \\ \tilde{C}_{\sigma_{k+1}} \end{bmatrix}\begin{bmatrix} E_{\sigma_k} \\ \tilde{C}_{\sigma_{k+1}} \end{bmatrix}^{\dagger}\right)\end{aligned} \tag{32}$$

with S_{σ_k} is an arbitrary matrix. Therefore, we have

$$T_{\sigma_k}=\Theta^{\dagger}_{\sigma_k}\lambda_1+S_{\sigma_k}\psi_{\sigma_k}\lambda_1,\ \forall\sigma_k\in\overline{1,N} \tag{33}$$

and

$$N_{\sigma_k}=\Theta^{\dagger}_{\sigma_k}\lambda_2+S_{\sigma_k}\psi_{\sigma_k}\lambda_2,\ \forall\sigma_k\in\overline{1,N} \tag{34}$$

Let Theorem 1 hold, the computed gains L_{σ_k} must satisfy the positivity of the matrix $T_{\sigma_k}\tilde{A}_{\sigma_k}-L_{\sigma_k}\tilde{C}_{\sigma_k}$ which is the essential condition to ensure that (14) becomes an interval observer for (12).

The fact that P_{σ_k} are both diagonal and positive definite matrices implies that $P_{\sigma_k}>0$ for all $\sigma_k\in\overline{1,N}$. In addition, $T_{\sigma_k}\tilde{A}_{\sigma_k}-L_{\sigma_k}\tilde{C}_{\sigma_k}\geq 0$ for all $\sigma_k\in\overline{1,N}$ then,

$$P_{\sigma_k}\left(T_{\sigma_k}\tilde{A}_{\sigma_k}-L_{\sigma_k}\tilde{C}_{\sigma_k}\right)\geq 0 \tag{35}$$

Replace (33) in (35) so (25) is verified.

Consider the multiple Lyapunov function

$$V(k)=e^T(k)P_{\sigma_k}e(k) \tag{36}$$

where P_{σ_k} are diagonal positive definite matrices for all $\sigma_k\in\overline{1,N}$. The increment of the Lyapunov function (36) is

$$\begin{aligned}\Delta V_{\sigma_k}(e)&=V_{\sigma_k}(e(k+1))-V_{\sigma_k}(e(k))\\ &=e^T(k+1)P_{\sigma_k}e(k+1)-e^T(k)P_{\sigma_k}e(k)\\ &=e^T(k)\Pi^T_{\sigma_k}P_{\sigma_k}\Pi_{\sigma_k}e(k)-e^T(k)P_{\sigma_k}e(k)\\ &+\delta^T(k)\Phi^T_{\sigma_k}P_{\sigma_k}\Phi_{\sigma_k}\delta(k)\\ &+e^T(k)\Pi^T_{\sigma_k}P_{\sigma_k}\Phi_{\sigma_k}\delta(k)\\ &+\delta^T(k)\Phi^T_{\sigma_k}P_{\sigma_k}\Pi_{\sigma_k}e(k)\end{aligned} \tag{37}$$

with

$$\Pi_{\sigma_k}=T_{\sigma_k}\tilde{A}_{\sigma_k}-L_{\sigma_k}\tilde{C}_{\sigma_k}$$

By adding to and subtracting from (37) the terms $\alpha e^T(k)P_{\sigma_k}e(k)$ and $\gamma^2\delta^T(k)\delta(k)$, we get

$$\begin{aligned}&\Delta V_{\sigma_k}(e(k))=e^T(k)\left[\Pi^T_{\sigma_k}P_{\sigma_k}\Pi_{\sigma_k}-(1-\alpha)P_{\sigma_k}\right]e(k)\\ &+\delta^T(k)\Phi^T_{\sigma_k}P_{\sigma_k}\Phi_{\sigma_k}\delta(k)\\ &+e^T(k)\Pi^T_{\sigma_k}P_{\sigma_k}\Phi_{\sigma_k}\delta(k)+\delta^T(k)\Phi^T_{\sigma_k}P_{\sigma_k}\Pi_{\sigma_k}e(k)\\ &-\alpha e^T(k)P_{\sigma_k}e(k)+\gamma^2\delta^T(k)\delta(k)-\gamma^2\delta^T(k)\delta(k)\end{aligned} \tag{38}$$

Therefore (38) can be rewritten as

$$\begin{aligned}\Delta V_{\sigma_k}(e(k)) &= \begin{bmatrix} e(k) & \delta(k) \end{bmatrix}^T \Lambda_{\sigma_k} \begin{bmatrix} e(k) & \delta(k) \end{bmatrix} \\ &\quad -\alpha e^T(k) P_{\sigma_k} e(k) + \gamma^2 \delta^T(k)\delta(k)\end{aligned} \tag{39}$$

where

$$\Lambda_{\sigma_k} = \begin{bmatrix} \Pi_{\sigma_k}^T P_{\sigma_k} \Pi_{\sigma_k} - (1-\alpha) P_{\sigma_k} & \Pi_{\sigma_k}^T P_{\sigma_k} \Phi_{\sigma_k} \\ \Phi_{\sigma_k}^T P_{\sigma_k} \Pi_{\sigma_k} & \Phi_{\sigma_k}^T P_{\sigma_k} \Phi_{\sigma_k} - \gamma^2 I \end{bmatrix} \tag{40}$$

which can be reformulated as

$$\begin{aligned}\Lambda_{\sigma_k} = &\begin{bmatrix} -(1-\alpha) P_{\sigma_k} & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + \\ &\begin{bmatrix} \Pi_{\sigma_k}^T \\ \Phi_{\sigma_k}^T \end{bmatrix} P_{\sigma_k} \begin{bmatrix} \Pi_{\sigma_k} & \Phi_{\sigma_k} \end{bmatrix}\end{aligned} \tag{41}$$

Using the Schur complement, (41) can be seen as

$$\begin{bmatrix} -(1-\alpha)P_{\sigma_k} & 0 & \star \\ 0 & -\gamma^2 I & \Phi_{\sigma_k}^T P_{\sigma_k} \\ P_{\sigma_k} T_{\sigma_k} \tilde{A}_{\sigma_k} - P_{\sigma_k} L_{\sigma_k} \tilde{C}_{\sigma_k} & P_{\sigma_k}\Phi_{\sigma_k} & -P_{\sigma_k} \end{bmatrix} \preccurlyeq 0, \tag{42}$$

Let equations (33)-(34) and (24) hold, so one can deduce (28). Hence, (5) is ensured such that

$$V_{\sigma(k)}(e(k+1)) - V_{\sigma_k}(e(k)) \leq -\alpha V_{\sigma_k}(e(k)) + \gamma^2 \|\delta(k)\|_2^2 \tag{43}$$

One highlights that the tightness of the interval width can be evaluated. Thus, the estimation accuracy can be assessed in the presence of disturbances and uncertainties. Let the inequality (43) holds for $k \in [k_0, k)$, which implies that

$$\begin{aligned}V_{\sigma_k}(e(k)) &< (1-\alpha)^{(k-k_0)} V_{\sigma_k}(e(k_0)) \\ &\quad + \sum_{m=0}^{k-k_0-1} (1-\alpha)^m \gamma^2 \parallel \delta(k) \parallel_2^2\end{aligned} \tag{44}$$

Using (4), (44) yields

$$e^T(k)e(k) \leq \frac{1}{\alpha_1}\left((1-\alpha)^{(k-k_0)} V_{\sigma_k}(e(k_0)) + \frac{\gamma^2}{\alpha} \parallel \delta(k) \parallel_\infty^2\right) \tag{45}$$

Consequently,

$$\|e(k)\| \leq \frac{1}{\sqrt{\alpha_1}}\left((1-\alpha)^{(k-k_0)} V_{\sigma_k}(e(k_0)) + \frac{\gamma^2}{\alpha} \parallel \delta(k) \parallel_\infty^2\right)^{\frac{1}{2}} \tag{46}$$

When $k \to \infty$,

$$\lim_{k \longrightarrow \infty} \parallel e(k) \parallel_2 < \frac{\gamma}{\sqrt{\alpha_1 \alpha}} \sup(\parallel \delta(k) \parallel_\infty) \tag{47}$$

which shows that the tightness of the interval error width is bounded by $\frac{\gamma}{\sqrt{\alpha_1\alpha}} \sup(\parallel \delta(k) \parallel_\infty)$. The latter depends only on γ for given α_1 and α. Consequently, optimizing γ improves the estimation accuracy.

By making a recursion for (5) over the interval $[k_l,\ k)$, we get

$$V_{\sigma_i}(e(k)) \leq (1-\alpha)^{k-k_l} V_{\sigma_i}(e(k_l)), \forall\ \sigma_i \in \overline{1, N} \tag{48}$$

Bearing in mind (6), then for all σ_i, $\sigma_j \in \overline{1, N}$, such that $\sigma_i \neq \sigma_j$

$$\begin{aligned}V_{\sigma_i}(e(k)) &\leq (1-\alpha)^{k-k_l} \frac{V_{\sigma_i}(e(k_l))}{V_{\sigma_j}(e(k_l))} V_{\sigma_j}(e(k_l)) \\ &\leq (1-\alpha)^{k-k_l} \frac{e^T(k_l) P_{\sigma_i} e(k_l)}{e^T(k_l) P_{\sigma_j} e(k_l)} e^T(k_l) P_{\sigma_j} e(k_l)\end{aligned} \tag{49}$$

Let (4) hold, then

$$\frac{e^T(k_l) P_{\sigma_i} e(k_l)}{e^T(k_l) P_{\sigma_j} e(k_l)} \leq \frac{\alpha_2}{\alpha_1} \tag{50}$$

Thereafter, (49) is equivalent to

$$V_{\sigma_i}(e(k)) \leq \frac{\alpha_2}{\alpha_1}(1-\alpha)^{k-k_l} V_{\sigma_j}(e(k_l)) \tag{51}$$

At switching time $k = k_l$, we have

$$\begin{aligned}V_{\sigma_i}(e(k_l)) &\leq \frac{\alpha_2}{\alpha_1} V_{\sigma_j}(e(k_l)) \\ &\leq \mu V_{\sigma_j}(e(k_l))\end{aligned} \tag{52}$$

As $\alpha_1 \leq \alpha_2$, it is trivial that $\mu = \frac{\alpha_2}{\alpha_1} > 1$.

Furthermore, the stability at the switching instants is guaranteed based on (6) which yields

$$\mu P_{\sigma_l} - P_{\sigma_k} \succeq 0 \tag{53}$$

By appying the Schur complement, we obtain

$$\begin{bmatrix} \mu P_{\sigma_l} & I \\ I & P_{\sigma_k}^{-1} \end{bmatrix} \succeq 0 \tag{54}$$

Let us multiply the both sides by $\begin{bmatrix} I & 0 \\ 0 & P_{\sigma_k} \end{bmatrix}$, we get (27) with $W_{\sigma_l} = \mu P_{\sigma_l}$.

An optimum average dwell time is fulfilled by defining an objective function added to the LMI conditions. This optimum is ensured by minimizing μ in the following objective function

$$\beta\mu + (1-\beta)\gamma,\ \beta \in [0\ 1] \tag{55}$$

The ISS conditions presented in Lemma 4 are verified for the estimation error $e = \overline{e} - \underline{e}$, hence one can deduce that (14) is an interval observer for (12). □

IV. NUMERICAL SIMULATIONS

Given the system (8) with two modes ($N = 2$) where

$$A_1 = \begin{bmatrix} 0.55 & 0.5 & 0.7 \\ 0 & 0.8 & 0.5 \\ 0 & 0 & 0.4 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 0.5 \\ 0.7 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0.2 & 0 \\ 0.2 & 0 & 0.2 \end{bmatrix},\ D_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.238 & -0.119 & 0.119 \\ 0 & 0.476 & 0.238 \\ 0 & 0 & 0.119 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.4 \\ 0.3 \\ 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.101 & 0 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{bmatrix},\ D_2 = \begin{bmatrix} 1 \\ 0 \\ 4.73 \end{bmatrix}$$

$w(k)$ and $v(k)$ are respectively the disturbances and the measurement noise which are uniformly bounded such that $\mid w(k) \mid \leq \overline{w}$ with $\overline{w} = \begin{bmatrix} 0.06 & 0.06 & 0.06 \end{bmatrix}$, and $\mid v(k) \mid \leq \overline{v}$ with $\overline{v} = \begin{bmatrix} 0.06 & 0.06 \end{bmatrix}$. The unknown input is given as $d(k) = 0.3\sin(0.5k)$.

To satisfy the assumptions of Theorems 1 and 2, the parameters T_{σ_k} and N_{σ_k} are determined as follows

$$T_1 = \begin{bmatrix} 0.7182 & -0.2182 & -0.2818 & 0 \\ -0.2765 & 0.2765 & -0.2765 & 0 \\ -0.9931 & 0.4931 & 0.0069 & 0 \\ -0.0388 & -0.4612 & -0.0388 & 0 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} 0.8249 & -0.1021 & -0.1744 & 0 \\ 0.0017 & 0.7896 & -0.0004 & 0 \\ -0.8286 & -0.443 & 0.1752 & 0 \\ -0.1771 & 0.1365 & -0.174 & 0 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 1.0910 & 1.4090 \\ 3.6175 & 1.3825 \\ -2.4654 & 4.9654 \\ 2.3061 & 0.1939 \end{bmatrix}, N_2 = \begin{bmatrix} 0.7229 & 1.021 \\ -2.1002 & 2.1039 \\ 3.8185 & 4.4296 \\ 3.1049 & -1.3652 \end{bmatrix},$$

For $\alpha = 0.9$ and $\alpha_1 = 0.1$, we get

$$P_1 = \begin{bmatrix} 0.1248 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.106 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}$$

The interval observer gains are computed by $L_{\sigma_k} = P_{\sigma_k}^{-1} G_{\sigma_k}$ with

$$L_1 = \begin{bmatrix} 0.9227 & 1.4046 \\ 0.4147 & -0.8295 \\ -0.5104 & -2.7310 \\ -1.9418 & -1.3664 \end{bmatrix}, L_2 = \begin{bmatrix} 1.5188 & -1.4676 \\ -2.5563 & 2.586 \\ -0.8413 & -1.1224 \\ -1.2693 & 0.8605 \end{bmatrix},$$

$\mu = 1.2476$ leads to an average dwell time $\tau_a > 0.0961$ and $\gamma = 4.189$. For the simulation, the switched signal σ_k verifying the average dwell time is plotted in Figure 1. The intervals that enclose the various components of the state by using the H_∞ norm are depicted in Figure 2 where solid and dashed lines represent respectively the state and the estimated bounds. The reconstruction of the unknown input $d(k)$ is drawn in Figure 3. The interval observer simultaneously returns both state and unknown input estimates as expected.

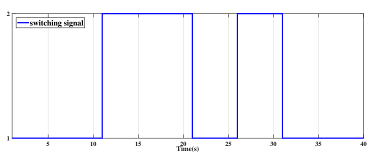

Fig. 1: The switching signal

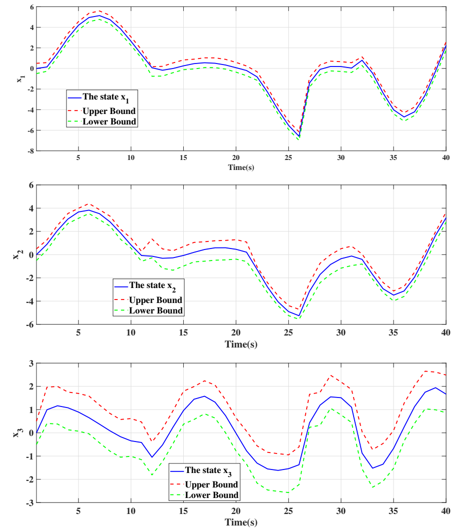

Fig. 2: State and estimated bounds

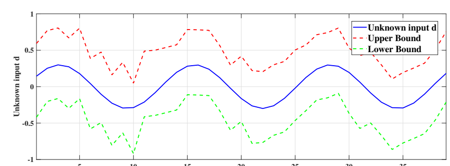

Fig. 3: The unknown input

V. Conclusion

A simultaneous input and state interval observer is investigated in this paper for uncertain discrete-time switched systems. The effect of unknown disturbances and measurement noise is also considered and an H_∞ formalism is employed to attenuate this effect. Interval observer based on the introduction of new matrices providing more degrees of freedom is designed. Simulation results illustrate the effectiveness of the present method. The L_∞ norm based design method in the same spirit and the extension to Linear-Parameter Varying switched systems are promising directions for future works.

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