Fast interval estimation for discrete-time linear systems: An L_1 optimization method

Zhenhua Wang, Thach Ngoc Dinh, Qinghua Zhang, Tarek Raissi, Yi Shen

Fast interval estimation for discrete-time linear systems: An L_1 optimization method ⋆

Zhenhua Wang [a], Thach Ngoc Dinh [b], Qinghua Zhang [c], Tarek Raïssi [b], Yi Shen [a]

[a] *School of Astronautics, Harbin Institute of Technology, Harbin 150001, P. R. China*

[b] *Conservatoire National des Arts et Métiers (CNAM), Cedric-Lab, Paris 75141, France*

[c] *Univ. Gustave Eiffel, Inria, COSYS-SII, I4S, 35042 Rennes, France*

Abstract

This paper studies interval estimation for discrete-time linear systems with unknown but bounded disturbances. Inspired by the parity space approach, we propose a point estimator with fixed-time convergence property. The estimator is combined with the zonotope-based interval analysis to achieve fast interval estimation. The parameter matrix in the estimator is optimized by minimizing the length of the edges of the outer box of the error zonotope. It is formulated as L_1 optimization problem and can be efficiently solved by linear programming. Comparison studies illustrate the superiority of the proposed method over existing techniques.

Key words: Interval estimation, zonotope, fixed-time convergence, L_1 optimization, linear programming.

1 Introduction

State estimation is important in many applications such as controller design and fault diagnosis. To attenuate the effect of uncertainties on state estimation, many robust estimation methods have been proposed (Kalman , 1960; Xie & Souza , 1993; Hammouri et al. , 2002). Kalman filtering and H_∞ observer design are two commonly used robust estimation techniques. Compared with the Gaussian noise assumption in Kalman filtering and the energy-bounded assumption in H_∞ observer design, an alternative assumption is that the uncertainties are unknown but bounded. Based on this assumption, interval estimation has been studied in the past decades (Gouze et al., 2000; Mazenc & Bernard, 2011; Raïssi et al., 2012; Wang et al., 2018).

Interval observer is a frequently used interval estimation method. Nevertheless, the design conditions of interval observers are usually restrictive. To relax them, coordinate transformations-based methods have been proposed in (Mazenc & Bernard, 2011; Raïssi et al., 2012). However, the coordinate transformations may cause large conservatisms. Recently, a direct design method based on a new interval observer structure has been proposed in (Wang et al., 2018). Compared with the basic interval observer, the interval observer proposed in Wang et al. (2018) has more degrees of design freedom, which can be optimized to improve the estimation accuracy.

Recently, fixed-time estimation has also attracted some attention (Engel & Kreisselmeier , 2002; Menard et al. , 2017; Rios & Teel , 2018). However, most existing results focus on fixed-time point estimation for continuous-time systems. To the best of our knowledge, only Zhang et al. (2019), Dinh et al. (2019), and Meslem & Ramdani (2020) have studied fixed-time interval estimation for discrete-time systems. However, optimal design of the estimators has not been considered in these papers.

Inspired by the parity space approach, a fixed-time interval estimation method has been proposed in Wang et al. (2020), where a Frobenius norm criterion is used to optimize the parameter matrix in the estimator. Although

⋆ This work was supported by National Natural Science Foundation of China (Grant No. 61973098, 61773145). This work is an extension of a paper presented at the 21st IFAC World Congress (Wang et al., 2020). Corresponding author: Zhenhua Wang. Tel. +86-451-86413411-8602.

Email addresses: zhenhua.wang@hit.edu.cn (Zhenhua Wang), ngoc-thach.dinh@lecnam.net (Thach Ngoc Dinh), qinghua.zhang@inria.fr (Qinghua Zhang), tarek.raissi@cnam.fr (Tarek Raïssi), shen@hit.edu.cn (Yi Shen).

the method in Wang et al. (2020) is optimal in the sense of Frobenius norm, it still has some conservatism. This paper proposes a new parameter optimization method based on the 1-norm criterion. With this optimization method, the length of the edges of the outer box of the error zonotope is minimized. The proposed design is formulated as a series of L_1 optimization problems, which can be efficiently solved by linear programming.

Notations. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ stand for n and $m \times n$ dimensional real Euclidean space, respectively. 0 is used to denote the zero matrix with appropriate dimensions, I_n denotes $n\times n$ identity matrix, and $\mathbf{1}_n$ is an n-dimensional vector with all entries equal to 1. For a vector $x \in \mathbb{R}^n$, its L_1-norm is defined as $\|x\|_1 = \sum_{i=1}^{n} |x_i|$. For a matrix M, M^+ and M^- are defined as $M^+ = \max(M, 0)$ and $M^- = M^+ - M$, respectively. Throughout this paper, the operators $\geq$, $\leq$, $|\cdot|$, and $\max(\cdot)$ on vectors and matrices should be understood element-wise.

2 Preliminaries and problem formulation

Definition 1. An m-order zonotope in the n-dimensional space is defined as

$$\mathcal{Z} = \langle p, H\rangle = \{p + Hz : z \in \mathbb{B}^m\}$$

where $p \in \mathbb{R}^n$ is the center of $\mathcal{Z}$, $H \in \mathbb{R}^{n\times m}$ is the shape matrix of $\mathcal{Z}$, and $\mathbb{B}^m = [-1, 1]^m$ is a hypercube.

Property 1 (Le et al., 2015). Consider $x \in \langle p, H\rangle \in \mathbb{R}^n$, the smallest axis-aligned box enclosing x is

$$p - \begin{bmatrix} \|H_1\|_1 \\ \vdots \\ \|H_n\|_1 \end{bmatrix} \leq x \leq p + \begin{bmatrix} \|H_1\|_1 \\ \vdots \\ \|H_n\|_1 \end{bmatrix} \tag{1}$$

where H_i denotes the ith row of the matrix H.

Consider the following system

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + D_1 d_k \\ y_k = Cx_k + D_2 d_k \end{cases} \tag{2}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state, $y_k \in \mathbb{R}^{n_y}$ is the measurement, $u_k \in \mathbb{R}^{n_u}$ is the known input, and $d_k \in \mathbb{R}^{n_d}$ is the unknown input. It is assumed that A is invertible, (C, A) is observable, and d_k is bounded as $|d_k| \leq \bar{d}$, which is equivalent to $d_k \in \langle 0, W_d\rangle$, where $W_d = \operatorname{diag}(\bar{d})$. Note that $\bar{d} \in \mathbb{R}^{n_d}$ is a known vector.

The aim of this paper is to achieve fast interval estimation of x_k, which consists of two aspects: First, in the disturbance-free situation, the estimation converges to the state in fixed time. Second, a box containing the true state can be obtained in the presence of unknown input.

3 Main results

Since A is invertible, the system in (2) is equivalent to the following *backward propagation representation*

$$\begin{cases} x_{k-1} = \tilde{A}x_k + \tilde{B}u_{k-1} + \tilde{D}_1 d_{k-1} \\ y_k = Cx_k + D_2 d_k \end{cases} \tag{3}$$

where $\tilde{A} = A^{-1}$, $\tilde{B} = -A^{-1}B$, and $\tilde{D}_1 = -A^{-1}D_1$.

Based on (3), it is easy to obtain the following relation

$$\mathbf{y}_k = M_x x_k + M_u \mathbf{u}_k + M_d \mathbf{d}_k \tag{4}$$

where

$$\mathbf{y}_k = \begin{bmatrix} y_k \\ \vdots \\ y_{k-s} \end{bmatrix}, \mathbf{u}_k = \begin{bmatrix} u_k \\ \vdots \\ u_{k-s} \end{bmatrix}, \mathbf{d}_k = \begin{bmatrix} d_k \\ \vdots \\ d_{k-s} \end{bmatrix},$$

$$M_x = \begin{bmatrix} C \\ C\tilde{A} \\ \vdots \\ C\tilde{A}^s \end{bmatrix}, M_u = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & C\tilde{B} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & C\tilde{A}^{s-1}\tilde{B} & \cdots & C\tilde{B} \end{bmatrix},$$

$$M_d = \begin{bmatrix} D_2 & 0 & \cdots & 0 \\ 0 & C\tilde{D}_1 + D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & C\tilde{A}^{s-1}\tilde{D}_1 & \cdots & C\tilde{D}_1 + D_2 \end{bmatrix}.$$

Herein, s is referred to as the order of the estimator.

Based on (4), we propose the following point estimator

$$\hat{x}_k = T(\mathbf{y}_k - M_u \mathbf{u}_k), \quad k \geq s \tag{5}$$

where $T \in \mathbb{R}^{n_x\times(s+1)n_y}$ should satisfy

$$TM_x = I_{n_x} \tag{6}$$

In the disturbance-free case, (4) becomes

$$\mathbf{y}_k = M_x x_k + M_u \mathbf{u}_k \tag{7}$$

Substituting (6) and (7) to (5), we have

$$\hat{x}_k = x_k, \quad k \geq s. \tag{8}$$

Therefore, $\hat{x}_k$ equals to x_k after s steps in the disturbance-free case.

In the presence of unknown input d_k, the state estimation $\hat{x}_k$ is not exactly equal to x_k. In this situation, we aim to estimate an axis-aligned box containing x_k.

By substituting (4) and (6) into (5), we have

$$\hat{x}_k = x_k + TM_d\mathbf{d}_k$$

which implies that

$$x_k \in \langle \hat{x}_k, H \rangle \tag{9}$$

with $H \in \mathbb{R}^{(s+1)n_x \times (s+1)n_d}$ given by

$$H = -TM_d\mathbf{W}_d = -TM_d \begin{bmatrix} W_d & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & W_d \end{bmatrix}.$$

Based on (9) and Property 1, we can obtain

$$\hat{x}_k - \begin{bmatrix} \|H_1\|_1 \\ \vdots \\ \|H_{n_x}\|_1 \end{bmatrix} \leq x_k \leq \hat{x}_k + \begin{bmatrix} \|H_1\|_1 \\ \vdots \\ \|H_{n_x}\|_1 \end{bmatrix}. \tag{10}$$

In (10), $H_i \in \mathbb{R}^{1\times(s+1)n_d}$ denotes the ith row of H.

From (10), it is known that the estimation accuracy can be measured by $\|H_1\|_1, \cdots, \|H_{n_x}\|_1$, which denote the length of the edges of the outer box of the error zonotope. A possible way to optimize the matrix T is minimizing $\|H_1\|_1, \cdots, \|H_{n_x}\|_1$. To this end, an L_1 optimization method is proposed in the paper.

Matrix T can be written in the following form

$$T = \begin{bmatrix} t_1 & \cdots & t_{n_x} \end{bmatrix}^T \tag{11}$$

Then, (6) and (10) become

$$\begin{cases} t_1^T M_x = e_1^T \\ \vdots \\ t_{n_x}^T M_x = e_{n_x}^T \end{cases} \tag{12}$$

$$\begin{cases} \hat{x}_{1,k} - \|H_1\|_1 \leq x_{1,k} \leq \hat{x}_{1,k} + \|H_1\|_1 \\ \vdots \\ \hat{x}_{n_x,k} - \|H_{n_x}\|_1 \leq x_{n_x,k} \leq \hat{x}_{n_x,k} + \|H_{n_x}\|_1 \end{cases} \tag{13}$$

where $e_i \in \mathbb{R}^{n_x}$ denotes the ith column of I_{n_x} and H_i has the following form

$$H_i = -t_i^T M_d \mathbf{W}_d \tag{14}$$

To make $\|H_i\|_1$ as small as possible, t_i should be designed by solving the following L_1 optimization problem

$$\begin{aligned} &\min_{t_i} \ \| -t_i^T M_d \mathbf{W}_d\|_1 \\ &\text{s.t. } t_i^T M_x = e_i^T \end{aligned} \tag{15}$$

The L_1 optimization in (15) is a basis pursuit problem (Tillmann , 2015). Motivated by the solution approach used in the basis pursuit, the following theorem is proposed to solve the problem in (15).

Theorem 1 *The solution to the L_1 optimization problem in* (15) *can be obtained by solving the following linear programming problem*

$$\begin{aligned} &\min_{t_i, z_1, z_2} \ \mathbf{1}^T_{(s+1)n_d}(z_1 + z_2) \\ &\text{s.t.} \begin{cases} M_x^T t_i = e_i \\ z_1 - z_2 = -(M_d\mathbf{W}_d)^T t_i \\ z_1 \geq 0 \\ z_2 \geq 0 \end{cases} \end{aligned} \tag{16}$$

PROOF. The problem in (15) is equivalent to

$$\begin{aligned} &\min_{t_i} \ \| -(M_d\mathbf{W}_d)^T t_i\|_1 \\ &\text{s.t. } M_x^T t_i = e_i \end{aligned} \tag{17}$$

By letting

$$z_1 = (-(M_d\mathbf{W}_d)^T t_i)^+, \ z_2 = (-(M_d\mathbf{W}_d)^T t_i)^- \tag{18}$$

we have

$$z_1 + z_2 = | -(M_d\mathbf{W}_d)^T t_i| \tag{19}$$

$$z_1 - z_2 = -(M_d\mathbf{W}_d)^T t_i \tag{20}$$

$$z_1 \geq 0 \tag{21}$$

$$z_2 \geq 0 \tag{22}$$

Based on the definition of 1-norm, we have

$$\| -(M_d\mathbf{W}_d)^T t_i\|_1 = \mathbf{1}^T_{(s+1)n_d} | -(M_d\mathbf{W}_d)^T t_i| \tag{23}$$

Substituting (19) in (23) yields

$$\| -(M_d\mathbf{W}_d)^T t_i\|_1 = \mathbf{1}^T_{(s+1)n_d}(z_1 + z_2) \tag{24}$$

Now the objective function is transformed to $\mathbf{1}^T_{(s+1)n_d}(z_1 + z_2)$. Then, by combining (20), (21), (22) and $M_x^T t_i = e_i$, the original problem in (15) can be transformed to the linear programming problem in (16). □

To analyze the influence of s on the performance of the proposed method, we propose the following theorem.

Theorem 2 *The accuracy of the interval estimation method does not decrease with increasing s.*

PROOF. We use T_s and T_{s+1} to denote the parameter matrices in sth order estimator and $(s+1)$th order estimator, respectively. Note that T_s is an $n_x \times (s+1)n_y$ dimensional matrix while T_{s+1} is an $n_x \times (s+2)n_y$ dimensional matrix. The $(s+1)$th order estimator has more parameters to be optimized than the sth order one.

Suppose that T_s^* is the optimal parameter matrix for the sth order estimator. If we choose $T_{s+1} = \begin{bmatrix} T_s^* & 0 \end{bmatrix}$, the $(s+1)$th order estimator reduces to the optimal sth order estimator. That is, the optimal sth order estimator is a special case of the $(s+1)$th order estimator. Therefore, the accuracy of the optimal $(s+1)$th order estimator is better or equal to that of the optimal sth order one, which implies that the accuracy of the proposed method does not decrease with increasing s. □

4 Simulation results

A numerical example adapted from Meslem & Ramdani (2020) is used to compare the proposed method with two existing approaches. The system has the form of (2) with the following parameters

$$A = \begin{bmatrix} 2.548 & -2.5165 & 0.9484 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0.8 \end{bmatrix},$$

$$x_0 = \begin{bmatrix} 3 & -2 & 0 \end{bmatrix}^T, u_k = \cos(100k^2), d_k \in [-1, 1]^5.$$

Considering $s = 2$ and using the proposed method, we obtain

$$T = \begin{bmatrix} 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

If the optimization method proposed in Wang et al. (2020) is used, we have

$$T = \begin{bmatrix} 0.9552 & 0.0634 & -0.2475 & -1.4044 & 0.1415 & -0.2133 \\ 0.1321 & 0.6788 & -0.001 & 0.0385 & -0.0068 & -0.0871 \\ -0.0235 & 0.0344 & 0.1696 & 0.689 & -0.0683 & 0.0457 \end{bmatrix}.$$

Using the method presented in Meslem & Ramdani (2020), we can obtain the following interval estimator

$$\begin{aligned} \hat{x}_k &= \mathcal{A}_y \begin{bmatrix} y_k \\ y_{k-1} \\ y_{k-2} \end{bmatrix} + \mathcal{A}_u \begin{bmatrix} u_k \\ u_{k-1} \\ u_{k-2} \end{bmatrix} \\ \bar{x}_k &= \hat{x}_k + \bar{\mathbf{w}} \\ \underline{x}_k &= \hat{x}_k + \underline{\mathbf{w}} \end{aligned}$$

where

$$\mathcal{A}_y = \begin{bmatrix} 0.3666 & -0.5056 & -0.618 & -0.5078 & 0.7847 & 0.3933 \\ -0.0463 & -0.0894 & -0.0053 & -0.0142 & 0.3648 & 0.1736 \\ -0.1706 & 0.1855 & 0.3568 & 0.2573 & 0.0068 & -0.0142 \end{bmatrix},$$

$$\mathcal{A}_u = \begin{bmatrix} 1.4401 & -1.5695 & 0 \\ -0.3353 & -0.7296 & 0 \\ -0.72 & 0.9864 & 0 \end{bmatrix}, \bar{\mathbf{w}} = \begin{bmatrix} 5.8269 \\ 2.1530 \\ 2.0907 \end{bmatrix}, \underline{\mathbf{w}} = -\bar{\mathbf{w}}.$$

The interval estimation results obtained by the above-mentioned methods are depicted in Figiures 1-3. Note that parameter optimization is not used in the method presented in Meslem & Ramdani (2020). As a result, both the optimization-based interval estimation methods provide more accurate estimation results than that presented in Meslem & Ramdani (2020). Figures 1-3 also show that the interval estimations obtained by the proposed method are more accurate than the method presented in Wang et al. (2020). Moreover, the volume of the error zonotope obtained by the proposed method is 3.584, while that obtained by the method in Wang et al. (2020) is 10.6128. This also shows the superiority of the proposed method.

In addition, the values of $\|H_1\|_1$, $\|H_2\|_1$, $\|H_3\|_1$ with different choices of s by the proposed method are given in Table 1. It can be seen that the accuracy of the interval estimation method does not decrease with increasing s.

Table 1
$\|H_1\|_1$, $\|H_2\|_1$, $\|H_3\|_1$ with different choices of s by the proposed method

	$s=2$	$s=3$	$s=4$
$\|H_1\|_1$	2.6	2.6	2.6
$\|H_2\|_1$	0.8	0.8	0.8
$\|H_3\|_1$	0.9	0.9	0.9

5 Conclusion

This paper proposes a fixed-time interval estimation method for discrete-time linear systems. The proposed

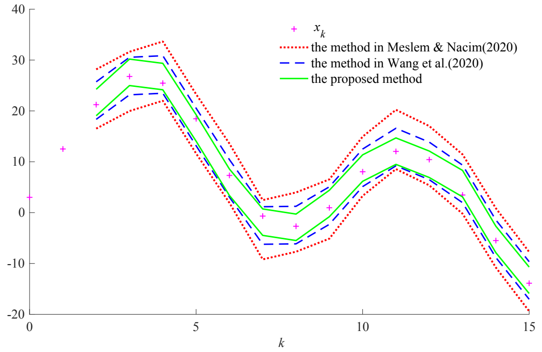

Fig. 1. The interval estimation results of $x_{1,k}$.

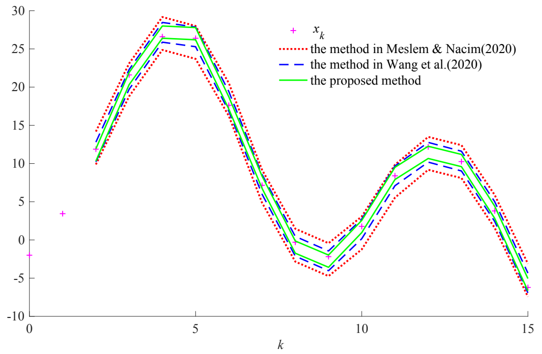

Fig. 2. The interval estimation results of $x_{2,k}$.

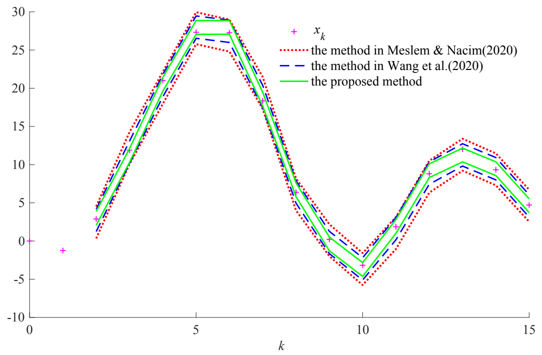

Fig. 3. The interval estimation results of $x_{3,k}$.

method is based on a point estimator and zonotope-based interval analysis. To improve the accuracy of estimation, an L_1 optimization method is proposed to design the parameter matrix in the estimator. The optimization problem can be converted to linear programming, which can be efficiently solved. Simulation results show that the proposed method can achieve more accurate interval estimation than two existing approaches.

References

R. E. Kalman. A new approach to linear filtering and prediction problems. *Journal of Basic Engineering*, 82: 35-45, 1960.

L. Xie, C. E. de Souza. H_∞ state estimation for linear periodic systems. *IEEE Transactions on Automatic Control*, 38: 1704-1707, 1993.

H. Hammouri, B. Targui, F. Armanet. High-gain observer based on a triangular structure. *International Journal of Robust and Nonlinear Control*, 12: 497-518, 2002.

J. L. Gouze, A. Rapaport, M. Z. Hadj-Sadok. Interval observers for uncertain biological systems. *Ecological Modelling*, 133: 45-56, 2000.

F. Mazenc, O. Bernard. Interval observer for linear time-invariant systems with disturbances. *Automatica*, 47: 140-147, 2011.

T. Raïssi, D. Efimov, A. Zolghadri. Interval state estimation for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 57: 260-265, 2012.

Z. Wang, C.-C. Lim, Y. Shen. Interval observer design for uncertain discrete-time linear systems. *Systems & Control Letters*, 116: 41-46, 2018.

R. Engel, G. Kreisselmeier. A continuous-time observer which converges in finite time. *IEEE Transactions on Automatic Control*, 47: 1202-1204, 2002.

T. Menard, E. Moulay, W. Perruquetti. Fixed-time observer with simple gains for uncertain systems. *Automatica*, 81: 438-446, 2017.

H. Rios , A. R. Teel. A hybrid fixed-time observer for state estimation of linear systems. *Automatica*, 87: 103-112, 2018.

J. Zhang, M. Chadli, Y. Wang. A fixed-time observer for discrete-time singular systems with unknown inputs. *Applied Mathematics and Computation*, 363: 124586, 2019.

T. Dinh, F. Mazenc , T. Raïssi. Finite-time guaranteed state estimation for discrete-time systems with disturbances. *The 4th conference on Control and Fault-Tolerant Systems*, Casablanca, Morocco, 2019.

N. Meslem, N. Ramdani. A new approach to design set-membership state estimators for discrete-time linear systems based on the observability matrix. *International Journal of Control*, 93: 2541-2550, 2020.

T. Menard, E. Moulay, W. Perruquetti. Fixed-time observer with simple gains for uncertain systems. *Automatica*, 81: 438-446, 2017.

Z. Wang, T. N. Dinh, Q. Zhang, T. Raïssi, Y. Shen. Fast interval estimation for discrete-time systems based on fixed-time convergence. *21st IFAC World Congress*, 2020.

V. T. H. Le, C. Stoica, T. Alamo, E. F. Camacho, D. Dumur. *Zonotopes: From Guaranteed State-estimation to Control*. ISTE Ltd and John Wiley & Sons, Inc., 2013.

A. M. Tillmann. Equivalence of linear programming and basis pursuit. *Proceedings in Applied Mathematics and Mechanics*, 15: 735-738, 2015.