POD associated with modal projection-based reduced order model for prestress structural vibrations

C. Hoareau, J.-F. Deü, R. Ohayon

Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 55-20 (2022) 169–174

POD associated with modal projection-based reduced order model for prestress structural vibrations

C. Hoareau *, **J.-F. Deü** * and **R. Ohayon** *

* *Laboratoire de Mécanique des Structures et des Systèmes Couplés, Conservatoire national des arts et métiers, Paris, France (e-mail: christophe.hoareau@lecnam.net).*

Abstract: The study deals with the computation of the linear vibrations of prestress elastic structures. The originality of the proposed approach is based on a Proper Orthogonal Decomposition (POD) associated with a modal projection-based Reduced Order Model (ROM) to evaluate the natural frequencies of an elastic structure in function of a static pressure load parameter. It is shown that a finite number of POD modes combined with linear prestress eigenmodes are able to evaluate efficiently the frequencies function of the evolution parameter. The generated reduced order model can be used in future works on parametric studies, sensitivity analyses, optimization procedure or feedback control loops systems.

Keywords: Geometrical nonlinearities, Follower forces, POD, Prestress modal analysis

1. INTRODUCTION

The prestress state is defined as the stress field acting in the structure when geometrical nonlinearities are taken into account in a static analysis. This paper is based on the theory of nonlinear elasticity, in which geometrical nonlinearities are considered with Saint-Venant Kirchhoff constitutive equations (see e.g. Ciarlet (1988), Belytschko et al. (2000), Wriggers (2008)). Concerning the dynamic analyses, it is shown in Ogden and Roxburgh (1993) and Herrmann (1956) that the stress has an influence on the natural frequencies of linear vibration analysis and the associated modal shape of the structure. Recently in Carrera et al. (2020), an efficient approach has been proposed to evaluate the influence of the prestress on the natural frequencies: (i) static nonlinear solutions are first computed in a given range of load leading to a change of the global stiffness of the system, (ii) then a linearized modal analysis is performed around each prestress state. However, this approach necessitates the computation of tangent stiffness matrices followed by an eigenvalue analysis at each increment of the parameter value. Moreover, for parametric studies, this procedure requires time and memory resources depending on the number of degrees of freedom.

The approach considered here is based on off-line and on-line steps. In the off-line phase, snapshots of the static solutions are selected to generate an *a posteriori* Proper Orthogonal Decomposition (POD) basis (see e.g. Berkooz et al. (1993)). This step is fundamental to obtain a Reduced Order Model (ROM) as seen in Radermacher and Reese (2016). It is shown here that the tangent stiffness matrix can be expressed as a sum of a finite number of matrices depending of the number of modes, which constitutes the originality of the present study. The feasibility and the efficiency of the methodology is analyzed through a numerical example involving non uniform follower forces. The numerical developments are based on the use of the open-source library FEniCS developed by Alnaes et al. (2015).

2. PRELIMINARY DEFINITIONS

Let Ω be an open bounded domain of $\mathbb{R}^3$, of boundary $\partial\Omega$. We denote by $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ the displacement vector field in Ω. From nonlinear elasticity theory, the Green-Lagrange strain tensor is defined by:

$$\boldsymbol{E}(\boldsymbol{u}) = \boldsymbol{\varepsilon}_L(\boldsymbol{u}) + \boldsymbol{\varepsilon}_Q(\boldsymbol{u}, \boldsymbol{u}) \tag{1}$$

where $\boldsymbol{\varepsilon}_L$ and $\boldsymbol{\varepsilon}_Q$ are given by:

$$\boldsymbol{\varepsilon}_L(\boldsymbol{u}) = \frac{1}{2}(\mathbf{Grad}^{\mathrm{T}}\boldsymbol{u} + \mathbf{Grad}\boldsymbol{u}) \tag{2}$$

$$\boldsymbol{\varepsilon}_Q(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2}(\mathbf{Grad}^{\mathrm{T}}\boldsymbol{u}\,\mathbf{Grad}\boldsymbol{v}) \tag{3}$$

The gradient operator **Grad** is defined in the reference configuration Ω. Using the second Piola-Kirchhoff stress tensor, the Saint-Venant Kirchhoff (SVK) constitutive equation is written as:

$$\boldsymbol{S}(\boldsymbol{u}) = \mathcal{D} : \boldsymbol{E}(\boldsymbol{u}) \tag{4}$$

where $\mathcal{D}$ is a constant fourth order tensor of elasticity for isotropic material defined by two constants.

3. EXPRESSION OF THE VIRTUAL WORK

3.1 *Virtual work principle*

The problem consists in finding $\boldsymbol{u}(\boldsymbol{X}, t)$ in $\mathcal{C}_u$ such that, $\forall\, \delta\boldsymbol{u}(\mathbf{X}) \in \mathcal{C}_u$ and for given initial conditions, we have:

$$\delta W_{\mathrm{acc}}(\ddot{\boldsymbol{u}}, \delta\boldsymbol{u}) + \delta W_{\mathrm{int}}(\boldsymbol{u}, \delta\boldsymbol{u}) + \delta W_{\mathrm{ext}}(\boldsymbol{u}, \delta\boldsymbol{u}) = 0 \tag{5}$$

in which $\delta\boldsymbol{u}$ is the virtual displacement field, $\mathbf{X}$ is the reference coordinate vector, t is the time and $\mathcal{C}_u$ is an admissible space of regular functions such that:

$$\mathcal{C}_u = \{\boldsymbol{u} \text{ “regular”} :\ \boldsymbol{u} = \mathbf{0} \in \Sigma_u\} \tag{6}$$

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2022.09.090

where Dirichlet boundary conditions are given on $\Sigma_u \subset \partial\Omega$. In Eq. (5), the virtual inertia work δW_{acc}, the virtual internal work δW_{int} and the virtual external work δW_{ext} are respectively given by:

$$\delta W_{\mathrm{acc}}(\ddot{\boldsymbol{u}}, \delta \boldsymbol{u}) = \int_\Omega \rho \ddot{\boldsymbol{u}} \cdot \delta \boldsymbol{u} \, d\Omega \tag{7}$$

$$\delta W_{\mathrm{int}}(\boldsymbol{u}, \delta \boldsymbol{u}) = \int_\Omega \delta \boldsymbol{E}(\boldsymbol{u}, \delta \boldsymbol{u}) : \boldsymbol{S}(\boldsymbol{u}) d\Omega \tag{8}$$

$$\delta W_{\mathrm{ext}}(p; \boldsymbol{u}, \delta \boldsymbol{u}) = \int_{\Sigma_p} p \, J(\boldsymbol{u}) \boldsymbol{F}^{-\mathrm{T}}(\boldsymbol{u}) \boldsymbol{n} \cdot \delta \boldsymbol{u} \, dS \tag{9}$$

where $\Sigma_p = \partial\Omega \backslash \Sigma_u$ is the part of the boundary where forces are prescribed and $\Sigma_p \cap \Sigma_u = \emptyset$. The virtual Green-Lagrange tensor is given by:

$$\delta \boldsymbol{E}(\boldsymbol{u}, \delta \boldsymbol{u}) = \boldsymbol{\varepsilon}_L(\delta \boldsymbol{u}) + \boldsymbol{\varepsilon}_S(\boldsymbol{u}, \delta \boldsymbol{u}) \tag{10}$$

where

$$\boldsymbol{\varepsilon}_S(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2}(\mathbf{Grad}^{\mathrm{T}} \boldsymbol{u} \, \mathbf{Grad} \boldsymbol{v} + \mathbf{Grad}^{\mathrm{T}} \boldsymbol{v} \, \mathbf{Grad} \boldsymbol{u}) \tag{11}$$

The follower forces are expressed considering known pressure distribution on the reference configuration $p(\boldsymbol{X})$. For more details the reader is referred to Hibbitt (1979). The deformation gradient is defined as $\boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{I} + \mathbf{Grad} \boldsymbol{u}$ and $J(\boldsymbol{u}) = \det(\boldsymbol{F}(\boldsymbol{u}))$. In the following, the virtual inertia work will also be denoted as the mass bi-linear operator m as:

$$m(\ddot{\boldsymbol{u}}, \delta \boldsymbol{u}) = \int_\Omega \rho \, \ddot{\boldsymbol{u}} \cdot \delta \boldsymbol{u} \, d\Omega \tag{12}$$

3.2 Virtual internal work as a polynomial cubic form of u

It can be shown that considering geometrical nonlinearities with a linear constitutive equation (SVK), the virtual internal work can be expressed as follows:

$$\delta W_{\mathrm{int}}(\boldsymbol{u}, \delta \boldsymbol{u}) = a_2(\boldsymbol{u}, \delta \boldsymbol{u}) + a_3(\boldsymbol{u}, \boldsymbol{u}, \delta \boldsymbol{u}) + a_4(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}, \delta \boldsymbol{u}) \tag{13}$$

where the forms a_2, a_3 and a_4 are given by:

$$a_2(\boldsymbol{u}, \delta \boldsymbol{u}) = \int_\Omega \boldsymbol{\varepsilon}_L(\delta \boldsymbol{u}) : \mathcal{D} : \boldsymbol{\varepsilon}_L(\boldsymbol{u}) \, d\Omega \tag{14}$$

$$\begin{aligned} a_3(\boldsymbol{u}, \boldsymbol{v}, \delta \boldsymbol{u}) = & \int_\Omega \boldsymbol{\varepsilon}_L(\delta \boldsymbol{u}) : \mathcal{D} : \boldsymbol{\varepsilon}_Q(\boldsymbol{u}, \boldsymbol{v}) d\Omega \\ & + \int_\Omega \boldsymbol{\varepsilon}_S(\delta \boldsymbol{u}, \boldsymbol{u}) : \mathcal{D} : \boldsymbol{\varepsilon}_L(\boldsymbol{v}) d\Omega \end{aligned} \tag{15}$$

$$a_4(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \delta \boldsymbol{u}) = \int_\Omega \boldsymbol{\varepsilon}_S(\delta \boldsymbol{u}, \boldsymbol{u}) : \mathcal{D} : \boldsymbol{\varepsilon}_Q(\boldsymbol{v}, \boldsymbol{w}) \, d\Omega \tag{16}$$

Let us note that all the forms are linear in terms of $\delta \boldsymbol{u}$. The term a_2 is linear in $\boldsymbol{u}$, a_3 is quadratic in $\boldsymbol{u}$ and a_4 is cubic in $\boldsymbol{u}$. Later we shall see that those forms are convenient to take into account the displacement as a linear combination of modes.

3.3 Virtual external work as a polynomial quadratic form of u

As seen in Eq. (9), we have considered in this paper the particular case of a pressure distribution which does not depend explicitly upon $\boldsymbol{u}$. The external work with those follower forces can be expressed as:

$$\begin{aligned} \delta W_{\mathrm{ext}}(p; \boldsymbol{u}, \delta \boldsymbol{u}) = & \, b_1(p; \delta \boldsymbol{u}) + b_2(p; \boldsymbol{u}, \delta \boldsymbol{u}) \\ & + b_3(p; \boldsymbol{u}, \boldsymbol{u}, \delta \boldsymbol{u}) \end{aligned} \tag{17}$$

where the forms b_1, b_2 and b_3 are all linear in $\delta \boldsymbol{u}$, b_2 is linear in $\boldsymbol{u}$ and b_3 is quadratic in $\boldsymbol{u}$ such that:

$$b_1(p; \delta \boldsymbol{u}) = \int_{\Sigma_p} p \, \boldsymbol{n} \cdot \delta \mathbf{u} \, dS \tag{18}$$

$$b_2(p; \boldsymbol{u}, \delta \boldsymbol{u}) = \int_{\Sigma_p} p \, \boldsymbol{G}(\mathbf{u}) \, \boldsymbol{n} \cdot \delta \mathbf{u} \, dS \tag{19}$$

$$b_3(p; \boldsymbol{u}, \boldsymbol{v}, \delta \boldsymbol{u}) = \int_{\Sigma_p} p \, \boldsymbol{H}(\boldsymbol{u}, \boldsymbol{v}) \, \boldsymbol{n} \cdot \delta \boldsymbol{u} \, dS \tag{20}$$

where the semi-column symbol is used to separate the known and unknowns fields. The matrices $\boldsymbol{G}$ and $\boldsymbol{H}$ are expressed as parts of the cofactor matrix of $\mathbf{F}$ (see footnotes [1]) defined by:

$$\mathbf{Cof}(\mathbf{F}(\boldsymbol{u})) = \mathbf{I} + \boldsymbol{G}(\boldsymbol{u}) + \boldsymbol{H}(\boldsymbol{u}, \boldsymbol{u}) \tag{23}$$

which gives the expression of $J\mathbf{F}^{-T} = \mathbf{Cof}(\mathbf{F})$ as seen in Ciarlet (1988).

4. STATIC AND DYNAMIC PROBLEMS

In the following we will consider vibration problems around nonlinear static equilibrium states (see Fig. 1).

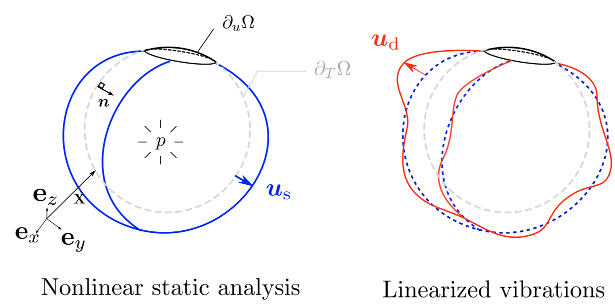

Fig. 1. Static and dynamic configurations.

The displacement solution $\boldsymbol{u}$ is then considered as a sum of a known static nonlinear displacement solution $\boldsymbol{u}_s$ and an unknown dynamic fluctuation of displacement $\boldsymbol{u}_{\mathrm{d}}$ such that:

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{s}} + \boldsymbol{u}_{\mathrm{d}} \tag{24}$$

The dynamic part of the displacement is supposed to be very small compared to a characteristic length of the structure l_c such that $\| \boldsymbol{u}_{\mathrm{d}} \| << l_c$. The linearized virtual works can be expressed as:

$$\delta W_{\mathrm{acc}}(\boldsymbol{u}, \delta \boldsymbol{u}) \simeq m(\ddot{\boldsymbol{u}}_{\mathrm{d}}, \delta \boldsymbol{u}) \tag{25}$$

$$\delta W_{\mathrm{int}}(\boldsymbol{u}, \delta \boldsymbol{u}) \simeq \delta W_{\mathrm{int}}(\boldsymbol{u}_{\mathrm{s}}, \delta \boldsymbol{u}) + k_{int}(\boldsymbol{u}_{\mathrm{s}}; \boldsymbol{u}_{\mathrm{d}}, \delta \boldsymbol{u}) \tag{26}$$

$$\begin{aligned} \delta W_{\mathrm{ext}}(p; \boldsymbol{u}, \delta \boldsymbol{u}) \simeq & \, \delta W_{\mathrm{ext}}(p; \boldsymbol{u}_{\mathrm{s}}, \delta \boldsymbol{u}) + k_{ext}(p, \boldsymbol{u}_{\mathrm{s}}; \boldsymbol{u}_{\mathrm{d}}, \delta \boldsymbol{u}) \\ & - f(\delta \boldsymbol{u}) \end{aligned} \tag{27}$$

in which the form $f(\delta \boldsymbol{u})$ is given by:

$$f(\delta \boldsymbol{u}) = \int_{\Sigma_p} \delta \boldsymbol{u} \cdot \boldsymbol{f} dS \tag{28}$$

[1] Expression of $\boldsymbol{G}(\boldsymbol{u})$ and $\boldsymbol{H}(\boldsymbol{u}, \boldsymbol{v})$ in Cartesian coordinate system:

$$\boldsymbol{G} = \begin{bmatrix} u_{3,Z} + u_{2,Y} & -u_{1,Y} & -u_{3,Z} \\ -u_{2,X} & u_{3,Z} + u_{1,X} & -u_{2,Z} \\ -u_{3,X} & -u_{3,Y} & u_{2,Y} + u_{1,X} \end{bmatrix} \tag{21}$$

$$\boldsymbol{H} = \begin{bmatrix} u_{2,Y} v_{3,Z} - u_{3,Y} v_{2,Z} & u_{3,Y} v_{1,Z} - u_{1,Y} v_{3,Z} & u_{1,Y} v_{2,Z} - u_{2,Y} v_{1,Z} \\ u_{2,Z} v_{3,X} - u_{3,Z} v_{2,X} & u_{3,Z} v_{3,Z} - u_{1,Z} v_{3,X} & u_{1,Z} v_{2,X} - u_{2,Z} v_{1,X} \\ u_{2,X} v_{3,Y} - u_{3,X} v_{2,Y} & u_{3,X} v_{1,Y} - u_{1,X} v_{3,Y} & u_{1,X} v_{2,Y} - u_{2,X} v_{1,Y} \end{bmatrix} \tag{22}$$

The known static field $\boldsymbol{u}_{\rm s}$ is the solution of:

$$\delta W_{\rm int}(\boldsymbol{u}_{\rm s}, \delta \boldsymbol{u}) + \delta W_{\rm ext}(p; \boldsymbol{u}_{\rm s}, \delta \boldsymbol{u}) = 0, \forall\, \delta \boldsymbol{u} \in \mathcal{C}_u \qquad (29)$$

The linearized dynamic problem around a prestress state is expressed as:

$$f(\delta \boldsymbol{u}) = m(\ddot{\boldsymbol{u}}_{\rm d}, \delta \boldsymbol{u}) + k_{\rm int}(\boldsymbol{u}_{\rm s}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) + k_{\rm ext}(p, \boldsymbol{u}_{\rm s}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}), \quad \forall\, \delta \boldsymbol{u} \in \mathcal{C}_u \qquad (30)$$

in which the form $k_{\rm int}$ is defined as follow:

$$k_{\rm int}(\boldsymbol{u}_{\rm s}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) = k_2(\boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) + k_3(\boldsymbol{u}_{\rm s}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) + k_4(\boldsymbol{u}_{\rm s}, \boldsymbol{u}_{\rm s}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) \qquad (31)$$

The various terms in the previous equation are given by:

$$k_2(\boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}_{\rm d}) = a_2(\boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) \qquad (32)$$

$$k_3(\boldsymbol{u}_{\rm s}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) = a_3(\boldsymbol{u}_{\rm d}, \boldsymbol{u}_{\rm s}, \delta \boldsymbol{u}) + a_3(\boldsymbol{u}_{\rm s}, \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) \qquad (33)$$

$$k_4(\boldsymbol{u}_{\rm s}, \boldsymbol{u}_{\rm s}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) = a_4(\boldsymbol{u}_{\rm d}, \boldsymbol{u}_{\rm s}, \boldsymbol{u}_{\rm s}, \delta \boldsymbol{u}) + a_4(\boldsymbol{u}_{\rm s}, \boldsymbol{u}_{\rm d}, \boldsymbol{u}_{\rm s}, \delta \boldsymbol{u}) + a_4(\boldsymbol{u}_{\rm s}, \boldsymbol{u}_{\rm s}, \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) \qquad (34)$$

in which all the forms are linear in $\boldsymbol{u}_{\rm d}$ and $\delta \boldsymbol{u}$. The forms k_3 and k_4 are respectively linear and quadratic in $\boldsymbol{u}_{\rm s}$. By considering the same procedure, the form $k_{\rm ext}$ is expressed as:

$$k_{\rm ext}(p, \boldsymbol{u}_{\rm s}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) = g_2(p; , \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) \qquad (35)$$

$$+ g_3(p; \boldsymbol{u}_{\rm s}, \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) \qquad (36)$$

where:

$$g_2(p; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) = b_2(p; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) \qquad (37)$$

$$g_3(p, \boldsymbol{u}_{\rm s}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) = b_3(p, \boldsymbol{u}_{\rm s}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) + b_3(p, \boldsymbol{u}_{\rm s}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) \qquad (38)$$

Those forms are implemented in practice in the open-source math-based front-end software FEniCS used for the numerical examples developed later in the manuscript.

5. REDUCED ORDER MODEL OF THE NONLINEAR STATIC PROBLEM

5.1 Reduced order model with POD modes

Let assume the existence of a POD basis denoted as $\mathcal{B}_{\rm POD} = \{\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_{\rm n}\}$ which is supposed to be known. One can express the static solution as:

$$u_{\rm s}(\boldsymbol{\mu}_{\rm s}; \boldsymbol{X}) \simeq \boldsymbol{u}_{\rm rom}(\boldsymbol{\mu}_{\rm s}; \boldsymbol{X}) = \sum_{i=1}^{\rm n} q_i(\boldsymbol{\mu}_{\rm s}) \boldsymbol{\varphi}_i(\boldsymbol{X}) \qquad (39)$$

where $\boldsymbol{\mu}_{\rm s}$ is a vector containing the parametric input of the problem (e.g. material, geometrical or load parameters). The unknowns of the problem are then the generalized coordinates q_i which depend on the parameter inputs. We will consider only a load distribution pressure p in this development so that $\boldsymbol{\mu}_{\rm s} = p_s$. Using Eq. (39) in the Eq. (29), and by taking successively $\delta \boldsymbol{u} = \boldsymbol{\varphi}_i$ as $\boldsymbol{\varphi}_i \in \mathcal{C}_u$, the reduced nonlinear problem can be expressed in the following matrix form:

$$[\mathbf{A} + \mathbf{B}(p)]\mathbf{q} + \mathbf{a}_{\rm nl}(\mathbf{q}) + \mathbf{b}_{\rm nl}(p; \mathbf{q}) = \mathbf{b}(p) \qquad (40)$$

where $\mathbf{q}$ is a vector containing the generalized coordinates. All the matrices and vectors are given by:

$$[\mathbf{A}]_{ij} = a_2(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i) \qquad (41)$$

$$[\mathbf{B}]_{ij} = b_2(p; \boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i) \qquad (42)$$

$$[\mathbf{a}_{\rm nl}]_i = \sum_{j=1}^{\rm n}\sum_{k=1}^{\rm n} q_j q_k a_3(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_k, \boldsymbol{\varphi}_i) + \sum_{j=1}^{\rm n}\sum_{k=1}^{\rm n}\sum_{l=1}^{\rm n} q_j q_k q_l a_4(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_k, \boldsymbol{\varphi}_l, \boldsymbol{\varphi}_i) \qquad (43)$$

$$[\mathbf{b}_{\rm nl}]_i = \sum_{j=1}^{\rm n}\sum_{k=1}^{\rm n} q_j q_k b_3(p; \boldsymbol{\varphi}_j, \boldsymbol{\varphi}_k, \boldsymbol{\varphi}_i) \qquad (44)$$

$$[\mathbf{b}]_i = b_1(p; \boldsymbol{\varphi}_i) \qquad (45)$$

This problem can be solved using a nonlinear solver as Newton-Raphson method if the number of modes remain reasonable. Indeed, one of the major drawback of this projection-based technique is due to the triple sum Eq. (43). A maximum of 10 POD modes will be considered in the numerical examples.

6. LINEAR VIBRATION PROBLEM AND MODAL PROJECTION BASIS

We consider now the linear vibration problem. Let us denote by ω the circular frequency. Taking the same notation $\boldsymbol{u}_{\rm d}$ for sake of brevity, the Eq. (30) in the time domain is written in the frequency domain as

$$k_{\rm int}(\boldsymbol{u}_{\rm rom}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) + k_{\rm ext}(p, \boldsymbol{u}_{\rm rom}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) - \omega^2 m(\boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) = f(\delta \boldsymbol{u}), \quad \forall\, \delta \boldsymbol{u} \in \mathcal{C}_u \qquad (46)$$

Given ω and f, the problem consists in finding $\boldsymbol{u}_{\rm d} \in \mathcal{C}_u$ such that Eq. (46) is satisfied $\forall\, \delta \mathbf{u} \in \mathcal{C}_u$. The associated eigenvalue problem consists in finding ω and $\boldsymbol{u}_{\rm d} \in \mathcal{C}_u$ such that:

$$k_{\rm int}(\boldsymbol{u}_{\rm rom}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) + k_{\rm ext}(p, \boldsymbol{u}_{\rm rom}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) - \omega^2 m(\boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) = 0, \quad \forall\, \delta \boldsymbol{u} \in \mathcal{C}_u \qquad (47)$$

Considering $\boldsymbol{u}_{\rm rom}$, the stiffness operators are written as:

$$k_{\rm int}(\boldsymbol{u}_{\rm rom}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) = k_2(\boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) + \sum_{i=1}^{\rm n} q_i k_3(\boldsymbol{\varphi}_i; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) + \sum_{i=1}^{\rm n}\sum_{j=1}^{\rm n} q_i q_j k_4(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) \qquad (48)$$

and

$$k_{\rm ext}(p, \boldsymbol{u}_{\rm s}; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) = g_2(p_{\rm s}; , \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) \sum_{i=1}^{\rm n} q_i g_3(p, \boldsymbol{\varphi}_i; \boldsymbol{u}_{\rm d}, \delta \boldsymbol{u}) \qquad (49)$$

Those equations above show expressions of the internal and external stiffness forms as a finite sum of operators which depends on the number n of POD modes. The number of operators involved is $({\rm n}+1)^2$ by counting the number of terms in Eqs. (48) and (49).

6.1 Finite element discretization

The finite element discretization of the left hand side of Eq. (30) leads to the eigenvalue problem given by:

$$[\mathbf{K}_{\rm tan}(p) - \omega^2 \mathbf{M}]\boldsymbol{\Psi} = \mathbf{0} \qquad (50)$$

where $\mathbf{K}_{\rm tan}$ is the tangent stiffness matrix (which depend on the follower force responsible for prestressing) and $\mathbf{M}$

is the mass matrix. The tangent stiffness matrix can be expressed as a finite sum of pre-computed operators:

$$\mathbf{K}_{\tan} = \mathbf{K}_2 + \mathbf{G}_2 + \sum_{i=1}^{n} q_i[\mathbf{K}_3^{(i)} + \mathbf{G}_3^{(i)}] + \sum_{i=1}^{n}\sum_{j=1}^{n} q_i q_j \mathbf{K}_4^{(ij)} \tag{51}$$

where the pre-computed operator are expressed off-line as a function of the POD basis $\mathcal{B}_{\text{POD}}$ detailed as:

$$\delta\mathbf{u}^{\mathrm{T}}\mathbf{K}_2\mathbf{u}_{\mathrm{d}} \Leftarrow k_2(\boldsymbol{u}_{\mathrm{d}}^{\mathrm{h}}, \delta\boldsymbol{u}^{\mathrm{h}}) \tag{52}$$

$$\delta\mathbf{u}^{\mathrm{T}}\mathbf{G}_2\mathbf{u}_{\mathrm{d}} \Leftarrow g_2(p; \boldsymbol{u}_{\mathrm{d}}^{\mathrm{h}}, \delta\boldsymbol{u}^{\mathrm{h}}) \tag{53}$$

$$\delta\mathbf{u}^{\mathrm{T}}\mathbf{K}_3^{(i)}\mathbf{u}_{\mathrm{d}} \Leftarrow k_3(\boldsymbol{\varphi}_i^{\mathrm{h}}; \boldsymbol{u}_{\mathrm{d}}^{\mathrm{h}}, \delta\boldsymbol{u}^{\mathrm{h}}) \tag{54}$$

$$\delta\mathbf{u}^{\mathrm{T}}\mathbf{G}_3^{(i)}\mathbf{u}_{\mathrm{d}} \Leftarrow g_3(p; \boldsymbol{\varphi}_i^{\mathrm{h}}; \boldsymbol{u}_{\mathrm{d}}^{\mathrm{h}}, \delta\boldsymbol{u}^{\mathrm{h}}) \tag{55}$$

$$\delta\mathbf{u}^{\mathrm{T}}\mathbf{K}_4^{(ij)}\mathbf{u}_{\mathrm{d}} \Leftarrow k_4(\boldsymbol{\varphi}_i^{\mathrm{h}}, \boldsymbol{\varphi}_j^{\mathrm{h}}; \boldsymbol{u}_{\mathrm{d}}^{\mathrm{h}}, \delta\boldsymbol{u}^{\mathrm{h}}) \tag{56}$$

where the superscript "h" stands for the finite element discretization of the fields. For a given value of pressure p, one can compute the eigenvalues and eigenvectors:

$$\{\omega_\beta^2, \boldsymbol{\psi}_\beta\}_{\beta=1..\mathrm{m}} \tag{57}$$

where $\omega_\beta = \omega_\beta(p)$ and $\boldsymbol{\psi}_\beta = \boldsymbol{\psi}_\beta(p)$ are resp. the circular eigenfrequencies and the eigenmode associated to the mode number β for a given pressure parameter p. For appropriate snapshots of static solutions, a called Prestress Modal Base (PMB) denoted $\mathcal{B}_{PMB}(p) = \{\psi_1(p), \ldots, \psi_\alpha(p)\}$ can be generated. Knowing the PMB, one can select appropriate vectors and generate a matrix denoted as $\mathbf{P}_{\text{PMB}}$ which contains all the selected vectors from those basis $\mathcal{B}_{PMB}(p)$. Finally, the projection of the pre-computed operators from Eqs. (52) to (56) used in Eq. (51) lead to a reduced eigenvalue problem.

$$[\mathbf{K}_{\mathrm{r}}(p) - \omega^2\mathbf{M}_{\mathrm{r}}]\mathbf{r} = \mathbf{0} \tag{58}$$

where $\mathbf{K}_{\mathrm{r}}$ is given by:

$$\mathbf{K}_{\mathrm{r}} = \mathbf{K}_{2\mathrm{r}} + \mathbf{G}_{2\mathrm{r}} + \sum_{i=1}^{n} q_i[\mathbf{K}_{3\mathrm{r}}^{(i)} + \mathbf{G}_{3\mathrm{r}}^{(i)}] + \sum_{i=1}^{n}\sum_{j=1}^{n} q_i q_j \mathbf{K}_{4\mathrm{r}}^{(ij)} \tag{59}$$

where all the operators with the subscript "r" are defined as:

$$[\quad]_{\mathrm{r}} = \mathbf{P}_{\text{PMB}}^{\mathrm{T}}[\quad]\mathbf{P}_{\text{PMB}} \tag{60}$$

7. NUMERICAL ANALYSIS

The numerical example has been implemented with the open-source finite element library FEniCS.

7.1 Computational methodology

The methodology is based on six steps illustrated in Fig. 2. *Step 1* consists in computing the nonlinear static solution on the parameter range. *Step 2* corresponds to the POD basis generation by performing a SVD on a snapshot matrix. The goal of *step 3* is to compute the operators from Eqs. (52) to (56). Then, *Step 4* consists in generating a modal prestress basis for given values of the pressure range and selecting a small number among them (the choice is out of scope of this paper). The *Step 5* corresponds to the projection of the operators already computed on the selected prestress modes. Those reduced operators are the reduced order model. Finally, *Step 6* consists in solving a reduced eigenvalue problem for a given parameter value.

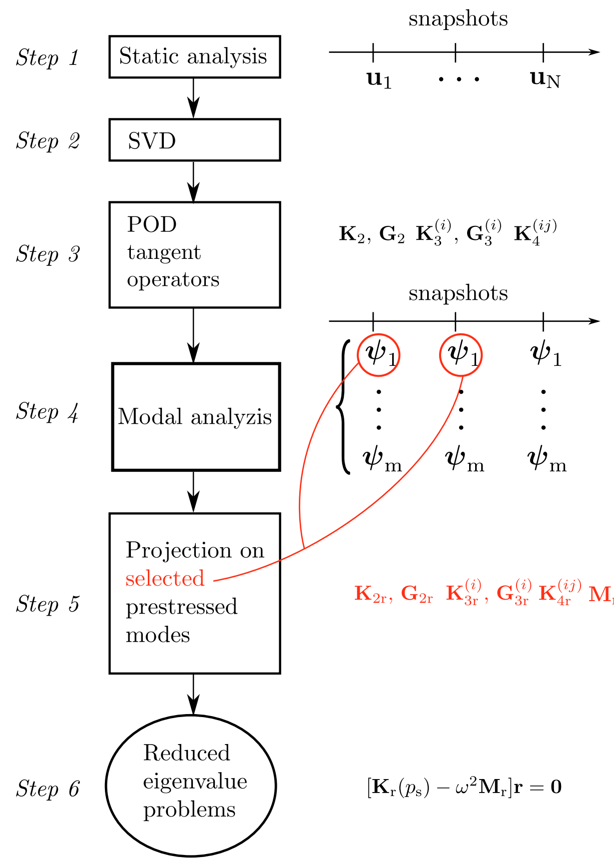

Fig. 2. Chart methodology.

The geometry of the structure considered for the numerical example is presented in Fig. 3. The structure is a beam of length $l = 1.0$ m, width $b = 0.1$ m and thickness $h = 0.01$ m. The beam is supposed to be homogeneous, isotropic, elastic and undergo large displacements and small strains. The Saint-Venant Kirchhoff material parameters are the Young modulus $E = 6 \times 10^7$ Pa and a Poisson ratio $\nu = 0.0$ (the surprising zero value of Poisson ratio has been chosen to obtain a circular deformation shape of the current configuration as seen in Fig. 4). The 3D beam is supposed to be clamped at one extremity and loaded by a linear follower force expressed as :

$$p(Z) = -\alpha Z \tag{61}$$

where $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ is an evolution parameter with $\alpha_{\min} = 0.0$ N.m^{-3} and $\alpha_{\max} = 4 \times 10^8$ N.m^{-3}.

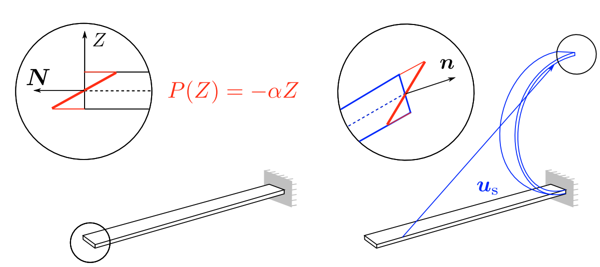

Fig. 3. Elastic 3D beams subjected to a non-uniform follower force: (left) reference configuration and (right) current configuration.

7.2 Static solutions and POD modes

In Fig. 4 shows snapshots of the solutions for two values of the pressure parameter. The solution is obviously nonlinear.

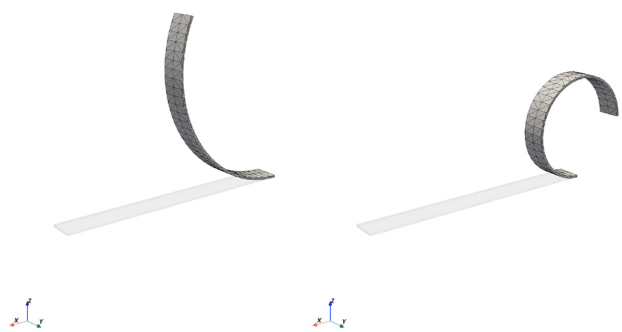

Fig. 4. Snapshots of the nonlinear static solution at $\alpha/\alpha_{\max} = [0.3; 0.7]$.

An SVD analysis has been done considering 100 snapshots of the direct nonlinear solutions. In Fig. 5 an illustration of the Singular Value Decomposition (SVD) of the snapshot matrix shows the extraction of the n first POD modes denoted as $\boldsymbol{\varphi}_i$.

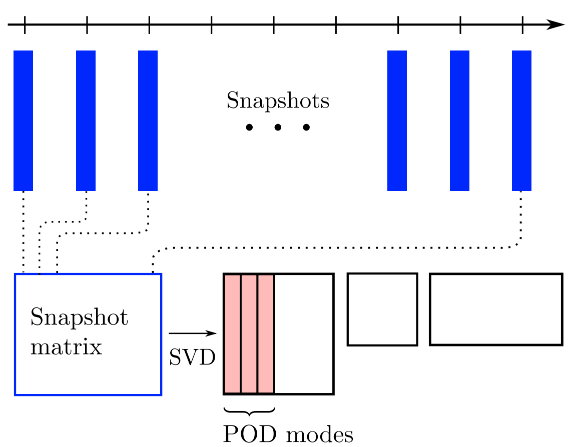

Fig. 5. Basis generation from snapshots and Singular Value Decomposition (SVD).

In Fig. 6 the POD modes and the associated generalized coordinates are presented.

The reconstruction of the solution from the nonlinear static reduced order model is presented in Fig. 7. The evolution of the displacement coordinates of the tip of the beam is plotted considering various number of POD modes. It can be observed that a small number of modes is needed to reconstruct the solution. In the following, ten POD modes are selected.

7.3 Modal analysis around a prestress state

An illustration of the linear prestress modal shape is given in Fig. 8. The mode is plotted around the static

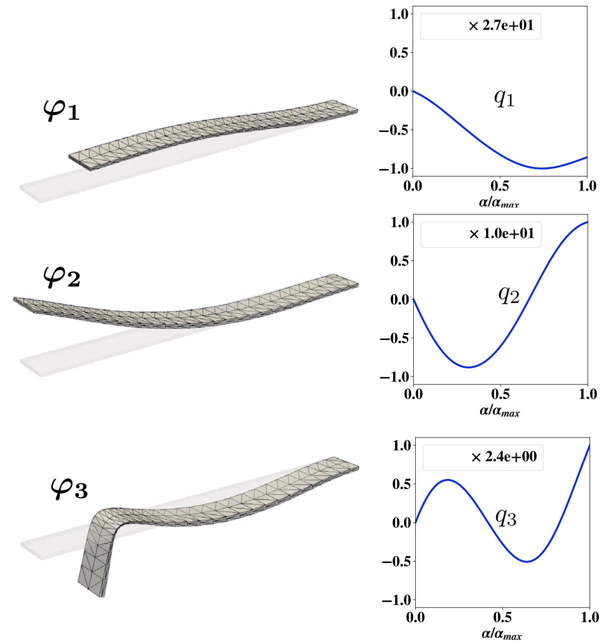

Fig. 6. POD modes $\boldsymbol{\varphi}_i$ and POD modal contributions q_i.

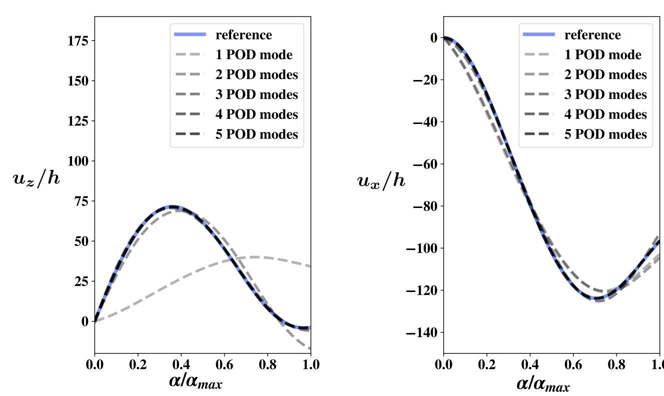

Fig. 7. Reconstruction of the dimensionless tip displacement.

nonlinear solution. The characterization of the modes are done considering a reciprocal mapping of the prestress mode on the reference configuration.

The Figures 9 and 10 show the evolution of the first five frequencies. A comparison between the solutions (in continuous line) from the classical approach and the ROM (in dashed line) are plotted. We recall that the number of POD modes is chosen equal to 10 for both figures. Fig. 9 shows the results considering 20 prestressed modes. The discrepancy between the two approaches remains important. Fig. 10 shows the results considering 55 prestress modes. The curves tend to be superposed when the number of selected modes increase.

8. CONCLUSION

The computation of the linearized vibrations of prestress elastic structures have been presented. An original approach consisting in associating a POD with a modal projection based reduced order model have been used. Eigenmodes of an elastic structure are computed as a

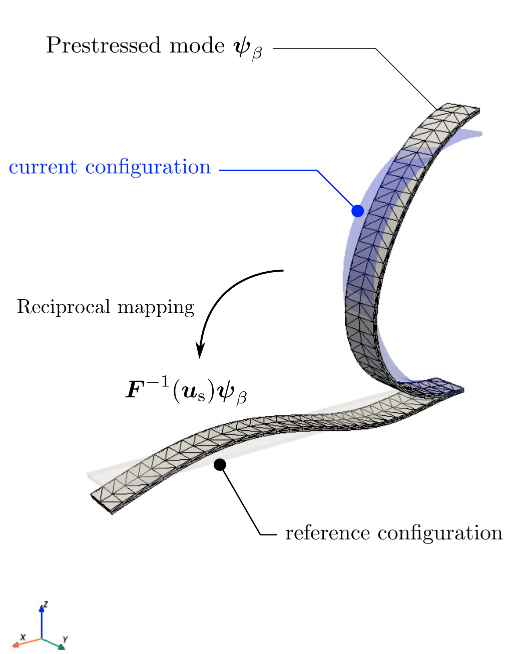

Fig. 8. Third flexural mode around the current configuration and its reciprocal mapping on the reference configuration.

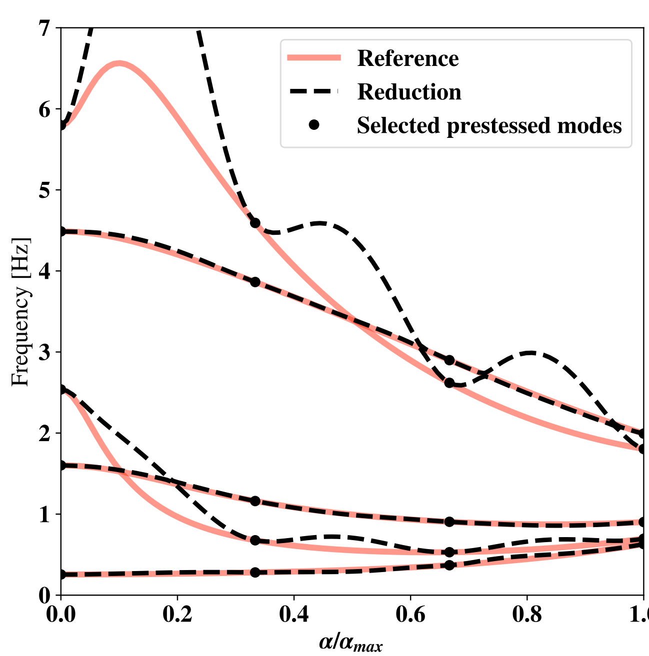

Fig. 9. Evolution of frequencies as a function of load parameter with 10 POD modes and 20 prestress modes.

function of a static pressure load parameter. The procedure shows an efficient estimation of the eigen-frequencies and the eigenmodes. Further studies to select a minimal number of appropriate modes are under investigation.

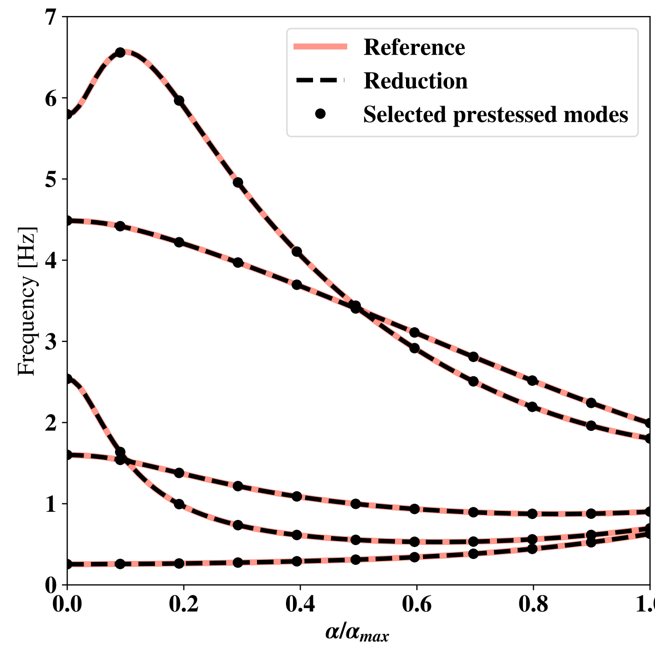

Fig. 10. Evolution of frequencies as a function of load parameter with 10 POD modes and 55 prestress modes.

REFERENCES

Alnaes, M.S., Hake, J., Johansson, A., Kehlet, B., Logg, A., Richardson, C., Ring, J., Rognes, M.E., and Wells, G.N. (2015). *Automated Solution of Differential Equations by the Finite Element Method.*

Belytschko, T., Liu, W., and Moran, B. (2000). *Nonlinear Finite Elements for Continua and Structures.* Wiley.

Berkooz, G., Holmes, P., and Lumley, J. (1993). The proper orthogonal decomposition in the analysis of turbulent flows. *Annu. Rev. Fluid Mech.*, 25, 539–75.

Carrera, E., Pagani, A., and Augello, R. (2020). Effect of large displacements on the linearized vibration of composite beams. *Int. J. Non. Linear Mech.*, 120, 103390.

Ciarlet, P.G. (1988). *Mathematical elasticity: Three-dimensional elasticity*, volume 84. SIAM.

Herrmann, G. (1956). The influence of initial stress on the dynamic behavior of elastic and viscoelastic plates. *Publ. Internat. Assoc. Bridge Struct. Eng.*, 16, 275–294.

Hibbitt, H.D. (1979). Some follower forces and load stiffness. *Int. J. Numer. Methods Eng.*, 14, 937–941.

Ogden, R. and Roxburgh, D. (1993). The effect of prestress on the vibration and stability of elastic plates. *Internat. J. Engrg. Sci.*, 31, 1611—1639.

Radermacher, A. and Reese, S. (2016). Pod-based model reduction with empirical interpolation applied to nonlinear elasticity. *Int. J. Numer. Methods Eng.*, 107(6), 477–495.

Wriggers, P. (2008). *Nonlinear finite element methods.* Springer.