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Abstract

In the context of damped second order linear dynamical systems, we study the asymptotic behavior of a time discretization of a slowly damped differential equation. We prove that this discretization can be constructed by means of a variable time step that gives rise to the same asymptotic behaviour as for the system in continuous time.

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1 Introduction

In this paper we study the asymptotics of an implicit discretization of the following differential equation on $\mathbb{R}^d$

$$\ddot{x}(t) + a(t)\dot{x}(t) + \nabla F(x(t)) = 0,\ t \geq 0. \tag{1}$$

Here a denotes a real-valued map that will principally satisfy the following assumptions: a will be positive, tending to 0 and merely $L^\infty(\mathbb{R}^+)$. This behaviour was referred to, in the literature, as a slowly decaying damping (see [6, 7, 13, 20, 5]). For example, one can take $a(t) := \frac{1}{(1+t)^\beta}$ for some $\beta \in (0,1)$. The function F is a real-valued function given on $\mathbb{R}^d$ which can be thought as a potential energy.

The asymptotic behaviour of solutions of (1) or some variants (e.g. a might also depend on $\dot{x}$) has been extensively studied. Indeed equations like (1) appear naturally as model of physical problems in finite and infinite dimensions. The asymptotic behaviour of such models reflects natural physical questions among which knowing the convergence to some equilibria or to specific sets is important. Let us note that in the framework of infinite dimension, equations like (1) appear while doing space discretizations of nonlinear damped wave equations.

Of course performing numerical simulations in order to illustrate or predict some asymptotic phenomena leads to understand if similar results that can be proved or are expected for models as (1) hold for discretization.

Systems like (1) are a particular case of dynamical systems of

$$\dot{x}(t) + G(x(t)) = 0 \tag{2}$$

for which the amount of works on asymptotics is far more huge. Examples occur of G that are gradients (one refers then to gradient systems) and for which non convergence to equilibrium points for (2) or some discretizations holds. One may quote [22] (see also [16]) in which an example of a smooth gradient system that has non convergent bounded trajectories is given and as a prolongation of this result, in contrast with the genericity given in [19], [14] gave an example of gradient system with an open set of initial data with non convergent bounded trajectories. For discretizations, a counterexample of convergence is given in [4].
Also in the context of gradient systems, S. Lojasiewicz [17, 18] showed the convergence of global and bounded solutions when the nonlinearity is analytic. The main ingredient of his proof is an inequality which relates the potential of the gradient (see (24)). This result was then generalised to the second order system (1), first in the case where $a(t) = 1$ (see [12]), then in the case where $a(t)$ tends to 0 (see [13]).

We consider below (see (7)) an implicit discretization of (1) mostly under the assumptions given in [13] where the asymptotic of (1) is studied.

Our main result (theorem 3.7) essentially says that the asymptotic behaviour of the sequence defined by this discretization is the same as the one given in the aforementioned paper. Precisely we give sufficient conditions in order to assert the existence of the sequence defining the discretization and the convergence of the discretized sequence to an equilibrium point. We would like to emphasize that, in order to obtain the decrease of some energy, we are lead to consider an implicit scheme in time with variable steps. The size of the steps are related to the assumptions on a but allows the time discretization to go to infinity.

We are also able to give the speed of convergence to the equilibrium and it is again the same as in the continuous case.

Let us also mention that the use of variable time steps in the discretization of (1) could also probably be helpful in an explicit or semi-implicit discretization of

$$\ddot{x}(t) + \|\dot{x}(t)\|^{\alpha}\dot{x}(t) + \nabla F(x(t)) = 0,\ t \geq 0, \tag{3}$$

but up until now we were only able to study the asymptotic behavior of an implicit scheme for (3). It should be emphasized that, for the implicit discretization of (3), in order to prove the convergence of bounded trajectories, in [15], a discrete angle condition obtained when F satisfies some Lojasiewicz inequality (see (24) below for the definition of such an inequality) is used. Such angle conditions first appeared in the continuous case and some specific discretizations in [1] (see also [9]), while for discrete situations they were also used in [2]. Despite their powerful potential, the results of this paper do not rely on them.

The following section presents the construction of our implicit discretization.

Then in section 3 we will state and prove our results of asymptotic convergence for our implicit scheme.

The last part of this paper consists of numerical simulations performed in situations with or without the theoretical assumptions of our main result.

2 Construction and existence of the variable time-step scheme

In this section we present the discretization that we are going to consider and prove its existence.
Let $F : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a C^1 function such that

$$\exists c_F > 0 /\ \forall u, v \in \mathbb{R}^d \quad \|\nabla F(u) - \nabla F(v)\| \leq c_F \|u - v\|. \tag{4}$$

It is easy to check that hypothesis (4) on F implies the following inequality (see for example [2, 15])

$$\forall u, v \in \mathbb{R}^d \quad < \nabla F(u) - \nabla F(v), u - v > \geq -c_F \|u - v\|^2, \tag{5}$$

$$\forall u, v \in \mathbb{R}^d \quad F(v) \geq F(u) + < \nabla F(u), v - u > -\frac{c_F}{2}\|u - v\|^2. \tag{6}$$

For (t_n) an increasing sequence of positive numbers and a a non-negative function, we consider a sequence $(u_n, v_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$\begin{cases} \dfrac{u_{n+1}-u_n}{\Delta t_n} = v_{n+1} \\ \dfrac{v_{n+1}-v_n}{\Delta t_n} = -a(t_{n+1})v_{n+1} - \nabla F(u_{n+1}) \\ u_0,\ v_0 \in \mathbb{R}^d \end{cases}, \tag{7}$$

where $\forall n \in \mathbb{N}$, $\Delta t_n := t_{n+1} - t_n$. The reason for which the sequence (Δt_n) is not assumed to be a constant as described in the introduction will appear in inequality (11).
The energy of the system governed by (7) is defined by

$$E(u,v) = \frac{1}{2}\|v\|^2 + F(u).$$

The existence and uniqueness of a sequence satisfying (7) is not clear in general. The proposition below gives some sufficient conditions for which it is the case.

Proposition 2.1. *Assume that F is of class $C^1(\mathbb{R}^d)$, coercive and that (4) holds. Let us also assume that $a \in L^\infty(\mathbb{R}^d)$ and that for any K compact subset of $\mathbb{R}$ there exists $\alpha_k > 0$ such that $a \geq \alpha_K$ on K, then for any $(u_0, v_0) \in \mathbb{R}^{2d}$, for any increasing sequence (t_n), if $\forall n \in \mathbb{N}$, $t_{n+1} - t_n$ is small enough the sequence (u_n, v_n) given by (7) is well defined, and we have*

$$\forall n \in \mathbb{N} \quad E(u_{n+1}, v_{n+1}) \leq E(u_n, v_n).$$

Proof. First of all, according to the continuity of F and its coercivity, we can assume that $F \geq 0$. We consider $(u_0, v_0) \in \mathbb{R}^{2d}$ and choose R such that if $E(u,v) \leq E_0 := \frac{1}{2}\|v_0\|^2 + F(u_0)$, then $u \in B(0,R)$. If we denote $R' = \sqrt{2E_0}$, we clearly have $\|v_0\| < R'$.
Let $\mathcal{N}$ be a bounded and continuous and globally lispchitz function coinciding with $-\nabla F$ on $B(0,3R)$. We denote $L_{F,R}$ the lipschtiz constant of $\mathcal{N}$.
We choose some $C > 0$ such that for all $n \in \mathbb{N}$, one has $\Delta t_n \leq C$.
Let us consider the following map defined on $\mathbb{R}^{2d}$:

$$\mathcal{F}(u,v) = (u_0 + \Delta t_0 v, v_0 - \Delta t_0 a(t_1)v + \Delta t_0 \mathcal{N}(u)). \tag{8}$$

We claim that for C small enough depending only on R, $\mathcal{F}$ is a contraction. Indeed, we have, for $((u,v),(u',v')) \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}$,

$$\|\mathcal{F}(u,v) - \mathcal{F}(u',v')\|_\infty \leq C(1 + |a|_\infty + L_{F,R})\|(u,v) - (u',v')\|_\infty,$$

which proves our claim.
This proves the existence of a unique (u_1, v_1) such that

$$(u_1, v_1) = \mathcal{F}(u_1, v_1).$$

Since $\|u_1\| \leq \Delta t_0 \|v_1\| + \|u_0\|$ and $\|v_1\|(1 + \Delta t_0 a(t_1)) \leq \|v_0\| + \Delta t_0 \|\mathcal{N}\|_\infty$, by the non negativeness of a, we can chose C small enough (depending only on R and R') so that $\|u_1\| < 2R$, so that for $n = 0$ one has

$$\begin{cases} \dfrac{u_{n+1} - u_n}{\Delta t_n} = v_{n+1} \\ \dfrac{v_{n+1} - v_n}{\Delta t_n} = -a(t_{n+1}) v_{n+1} - \nabla F(u_{n+1}). \end{cases}$$

Now by taking the scalar product of the second relation of (7) with $\Delta t_n v_{n+1}$, it comes

$$< \frac{v_{n+1} - v_n}{\Delta t_n}, \Delta t_n v_{n+1} >= -a(t_{n+1}) \Delta t_n \|v_{n+1}\|^2 - < \nabla F(u_{n+1}), \Delta t_n v_{n+1} >$$

or

$$\|v_{n+1}\|^2 - < v_n, v_{n+1} >= -a(t_{n+1}) \Delta t_n \|v_{n+1}\|^2 - < \nabla F(u_{n+1}), u_{n+1} - u_n > . \qquad (9)$$

By using the Cauchy-Schwarz inequality, there holds

$$- < v_n, v_{n+1} > \geq -\frac{1}{2} \|v_{n+1}\|^2 - \frac{1}{2} \|v_n\|^2. \qquad (10)$$

Combining (9) and (10), if follows that

$$\frac{1}{2} \|v_{n+1}\|^2 - \frac{1}{2} \|v_n\|^2 \leq -a(t_{n+1}) \Delta t_n \|v_{n+1}\|^2 - < \nabla F(u_{n+1}), u_{n+1} - u_n >,$$

and then

$$E(u_{n+1}, v_{n+1}) - E(u_n, v_n) \leq -a(t_{n+1}) \Delta t_n \|v_{n+1}\|^2 + F(u_{n+1}) - F(u_n) - < \nabla F(u_{n+1}), u_{n+1} - u_n > .$$

By using (6), we get

$$\begin{aligned} E(u_{n+1}, v_{n+1}) - E(u_n, v_n) &\leq -a(t_{n+1}) \Delta t_n \|v_{n+1}\|^2 + \frac{c_F}{2} \|u_{n+1} - u_n\|^2 \\ &= -\Delta t_n \left[a(t_{n+1}) - \frac{c_F}{2} \Delta t_n \right] \|v_{n+1}\|^2, \end{aligned} \qquad (11)$$

so that if Δt_n is small enough $E_{n+1} \leq E_n$ because our assumption on a implies that $\liminf_{t \to t_n} a(t) > 0$.

Since then $u_{n+1} \in B(0, R)$ and $v_{n+1} \in B(0, R')$ we can conclude by induction. □

Remark 2.2. It is clear from the proof that proposition 2.1 remains true if instead of (4), ∇F is only assumed to be locally lipschitz.

Remark 2.3. If we only assume that $a \in L^\infty(\mathbb{R})$ and that $a \geq 0$ *a.e.* we can still conclude that $\forall (u_0, v_0)$, there exists a non-decreasing sequence (t_n) such that the sequence (u_n, v_n) defined by (7) is well-defined up to some n such that $a(t_{n+1}) = 0$.

Remark 2.4. In the sequel, we will focus on the asymptotics of a sequence (u_n, v_n) satisfying (7). It is though not clear with the mere assumptions made on a that we have $t_n \to \infty$. In order to get this, specific a and sequences (t_n) such that $t_n \to \infty$ for which proposition 2.1 is true will be considered.

From now on, a will be a positive function in $L^\infty(\mathbb{R})$ such that

$$\exists c > 0,\ \beta \in (0,1)/\ \forall t \geq 0 \quad a(t) \geq \frac{c}{(1+t)^\beta}. \tag{12}$$

We may assume without loss of generality that

$$\forall t \in \mathbb{R} \quad a(t) \leq 1. \tag{13}$$

This choice of a is motivated by the work [13] in the continuous case.
We will now choose the sequence (t_n) as follows :

$$t_n = c_1 n^\gamma, \quad \gamma \in (\frac{1}{2}, 1),\ c_1 > 0. \tag{14}$$

Let us remark that

$$\Delta t_n := t_{n+1} - t_n = \frac{\gamma c_1}{n^{1-\gamma}} + O(\frac{1}{n^{2-\gamma}}) \sim \frac{\gamma c_1}{n^{1-\gamma}}. \tag{15}$$

We choose γ and c_1 such that

$$\gamma\beta = 1 - \gamma, \quad \frac{c}{c_1^\beta} - \frac{c_F}{2}\gamma c_1 > 0. \tag{16}$$

With these choices, there are two positive constants δ_1, δ_2 such that for all $n \in \mathbb{N}$ we have :

$$\delta_1 \Delta t_n \leq \frac{1}{(1+t_{n+1})^\beta} \leq \frac{1}{(1+t_n)^\beta} \leq \delta_2 \Delta t_n. \tag{17}$$

Note also that, with this choice, there exists $\eta > 0$ such that for n large enough

$$a(t_{n+1}) - \frac{1}{2} c_F \Delta t_n \geq \eta \Delta t_n,$$

and thus this proves, by (11),

Theorem 2.5. *For (t_n) defined by* (14)*, if F satisfies the same assumptions as in proposition 2.1, then for any $(u_0, v_0) \in \mathbb{R}^{2d}$, if $c_1 > 0$ is small enough, the sequence given by* (7) *is well defined. Moreover there exists $\eta > 0$ such that for all n large enough we have*

$$E(u_{n+1}, v_{n+1}) - E(u_n, v_n) \leq -\eta (\Delta t_n)^2 \|v_{n+1}\|^2. \tag{18}$$

Let us remark that with t_n defined by (14), we have $t_n \to \infty$. This will allow us to consider the asymptotic behavior of the sequence defined by (7). This will be done in the next section.

3 Asymptotics for the sequence (u_n, v_n).

In order to study the asymptotics of a sequence (u_n, v_n), we define its $\omega-$limit set

$$\omega((u_n, v_n)_{n\in\mathbb{N}}) = \{(a,b) \in \mathbb{R}^d \times \mathbb{R}^d \ : \exists n_k \to \infty / (u_{n_k}, v_{n_k}) \longrightarrow (a,b)\}.$$

Proposition 3.1. *Assume γ, c_1 as in (16) and F satisfies (4). Let (u_n, v_n) be a sequence satisfying (7)* [1]*. Assume also that F is bounded from below. Then the following assertions are true:*

1. $\lim\limits_{n\to+\infty} E(u_n, v_n)$ *exists.*

2. $\sum (\Delta t_n)^2 \|v_{n+1}\|^2$ *converges.*

Moreover, if we assume that (u_n) is bounded, then we get

3. *The set $\omega((u_n, v_n)_{n\in\mathbb{N}})$ is a nonempty compact connected subset of $\mathbb{R}^d \times \mathbb{R}^d$.*

4. *The function E is constant on $\omega((u_n, v_n)_{n\in\mathbb{N}})$.*

5. *There is a subsequence (v_{n_k}) such that $v_{n_k} \longrightarrow 0$.*

Proof. According to the theorem 2.5, $(E(u_n, v_n))$ is non increasing, thus converges in $\mathbb{R} \cup \{-\infty\}$. For F is bounded from below, we deduce that $(E(u_n, v_n))$ converges to a real number. Using again the theorem 2.5, $\sum (\Delta t_n)^2 \|v_{n+1}\|^2$ converges. Now as (u_n, v_n) is bounded ((v_n) is bounded since $(E(u_n, v_n))$ is non increasing and F is bounded from below), the set $\omega((u_n, v_n)_{n\in\mathbb{N}})$ is compact in $\mathbb{R}^d \times \mathbb{R}^d$. Besides, from the first relation of (7), one deduces that $(u_{n+1} - u_n)$ tends to 0. On the other hand, from the second relation of (7), we get

$$v_{n+1} - v_n = -a(t_{n+1})\Delta t_n v_{n+1} - \Delta t_n \nabla F(u_{n+1}) \underset{n\to+\infty}{\longrightarrow} 0.$$

It is therefore standard to prove that $\omega((u_n, v_n)_{n\in\mathbb{N}})$ is a connected part of $\mathbb{R}^d \times \mathbb{R}^d$.
Let $(a,b) \in \omega((u_n, v_n)_{n\in\mathbb{N}})$, then there exists a sequence (u_{n_k}, v_{n_k}) that converges to (a,b). Since E is continuous, then $(E(u_{n_k}, v_{n_k}))$ converges to $E(a,b)$. This ends the proof of the point 4 by using the point 1.
Now we prove the point 5. If it is not true, then

$$\exists \varepsilon > 0 \ \ \exists N \in \mathbb{N} \ \ \forall n \geq N \quad \|v_n\| > \varepsilon.$$

This contradicts the point 2 because $\sum (\Delta t_n)^2$ diverges. □

[1] Recall that if moreover F is coercive, there exists a sequence (u_n, v_n) satisfying (7).

Theorem 3.2. *Under the hypotheses of the proposition 3.1, let* $\mathcal{S} = \{a \in \mathbb{R}^d \ / \ \nabla F(a) = 0\}$ *and assume that*

$$\emptyset \neq \operatorname{argmin} F = \mathcal{S}. \tag{19}$$

Let (u_n, v_n) *be a sequence satisfying* (7) *and assume that* (u_n) *is bounded. Then*

$$\lim_{n\to+\infty} \|v_n\| + \|\nabla F(u_n)\| = 0.$$

Remark 3.3. Let us remark that this proves that for all $(a, b) \in \omega((u_n, v_n))$, $a \in \mathcal{S}$ and $b = 0$.

Proof. Now let ε be a positive real, and we define for all $n \in \mathbb{N}$

$$\Phi_\varepsilon(u_n, v_n) = E(u_n, v_n) + \frac{\varepsilon}{(1+t_n)^\beta} < \nabla F(u_n), v_n > .$$

Lemma 3.4. *There are two positives constants* ε_1, δ *and an integer* $N_1 \in \mathbb{N}$ *such that for all* $n \geq N_1$

$$\Phi_{\varepsilon_1}(u_n, v_n) - \Phi_{\varepsilon_1}(u_{n+1}, v_{n+1}) \geq \delta(\Delta t_n)^2[\|v_{n+1}\| + \|\nabla F(u_{n+1})\|]^2. \tag{20}$$

Proof. According to the theorem 2.5, for all $n \in \mathbb{N}$:

$$\begin{aligned} & \Phi_\varepsilon(u_{n+1}, v_{n+1}) - \Phi_\varepsilon(u_n, v_n) \\ \leq \ & -\eta(\Delta t_n)^2\|v_{n+1}\|^2 + \frac{\varepsilon}{(1+t_{n+1})^\beta} < \nabla F(u_{n+1}), v_{n+1} > - \frac{\varepsilon}{(1+t_n)^\beta} < \nabla F(u_n), v_n > . \end{aligned}$$

Now it is easy to see:

$$\begin{aligned} & \frac{\varepsilon}{(1+t_{n+1})^\beta} < \nabla F(u_{n+1}), v_{n+1} > \\ = \ & \frac{\varepsilon}{(1+t_{n+1})^\beta} < \nabla F(u_{n+1}), v_n - \Delta t_n a(t_{n+1}) v_{n+1} - \Delta t_n \nabla F(u_{n+1}) > \ \text{(by (7))} \\ = \ & -\frac{\varepsilon \Delta t_n}{(1+t_{n+1})^\beta}\|\nabla F(u_{n+1})\|^2 + \frac{\varepsilon}{(1+t_{n+1})^\beta} < \nabla F(u_{n+1}), v_n > \\ & \quad -\frac{\varepsilon \Delta t_n a(t_{n+1})}{(1+t_{n+1})^\beta} < \nabla F(u_{n+1}), v_{n+1} > \\ = \ & -\frac{\varepsilon \Delta t_n}{(1+t_{n+1})^\beta}\|\nabla F(u_{n+1})\|^2 + \frac{\varepsilon}{(1+t_{n+1})^\beta} < \nabla F(u_{n+1}), v_n > \\ & \quad +\frac{\varepsilon \Delta t_n a(t_{n+1})}{(1+t_{n+1})^\beta}\|\nabla F(u_{n+1})\|\|v_{n+1}\|, \end{aligned}$$

$$
\begin{aligned}
& -\frac{\varepsilon}{(1+t_n)^\beta} < \nabla F(u_n), v_n > \\
= \quad & -\frac{\varepsilon}{(1+t_n)^\beta} < \nabla F(u_n) - \nabla F(u_{n+1}) + \nabla F(u_{n+1}), v_n > \\
\leq \quad & \frac{\varepsilon}{(1+t_n)^\beta} \|\nabla F(u_n) - \nabla F(u_{n+1})\| \|v_n\| - \frac{\varepsilon}{(1+t_n)^\beta} < \nabla F(u_{n+1}), v_n > \\
\leq \quad & \frac{\varepsilon c_F}{(1+t_n)^\beta} \|u_n - u_{n+1}\| \|v_n\| - \frac{\varepsilon}{(1+t_n)^\beta} < \nabla F(u_{n+1}), v_n > \text{ by (4)} \\
\leq \quad & \frac{\varepsilon c_F \Delta t_n}{(1+t_n)^\beta} \|v_{n+1}\| \|v_{n+1} + \Delta t_n a(t_{n+1}) v_{n+1} + \Delta t_n \nabla F(u_{n+1})\| \\
& \qquad -\frac{\varepsilon}{(1+t_n)^\beta} < \nabla F(u_{n+1}), v_n > \text{ (by (7))} \\
\leq \quad & \frac{\varepsilon c_F \Delta t_n}{(1+t_n)^\beta} \|v_{n+1}\|^2 + \frac{\varepsilon c_F (\Delta t_n)^2 a(t_{n+1})}{(1+t_n)^\beta} \|v_{n+1}\|^2 + \frac{\varepsilon c_F (\Delta t_n)^2}{(1+t_n)^\beta} \|v_{n+1}\| \|\nabla F(u_{n+1})\| \\
& \qquad -\frac{\varepsilon}{(1+t_n)^\beta} < \nabla F(u_{n+1}), v_n > .
\end{aligned}
$$

Note that for n large enough, we have

$$
\begin{aligned}
& \frac{1}{(1+t_{n+1})^\beta} - \frac{1}{(1+t_n)^\beta} \\
= \quad & \frac{1}{(1+c_1(n+1)^\gamma)^\beta} - \frac{1}{(1+c_1 n^\gamma)^\beta} \\
= \quad & \frac{1}{c_1^\beta n^{\gamma\beta}} \frac{1}{\left(\frac{1}{c_1 n^\gamma} + \left(1+\frac{1}{n}\right)^\gamma\right)^\beta} - \frac{1}{c_1^\beta n^{\gamma\beta}} \frac{1}{\left(\frac{1}{c_1 n^\gamma} + 1\right)^\beta} \\
= \quad & \frac{1}{c_1^\beta n^{\gamma\beta}} \frac{1}{\left(\frac{1}{c_1 n^\gamma} + 1 + \frac{\gamma}{n} + O(\frac{1}{n^2})\right)^\beta} - \frac{1}{c_1^\beta n^{\gamma\beta}} \frac{1}{\left(\frac{1}{c_1 n^\gamma} + 1\right)^\beta} \\
\underset{\text{since } (1<2\gamma<2)}{=} \quad & \frac{1}{c_1^\beta n^{\gamma\beta}} \left[1 - \frac{\beta}{c_1 n^\gamma} - \frac{\beta\gamma}{n} + O(\frac{1}{n^{2\gamma}})\right] - \frac{1}{c_1^\beta n^{\gamma\beta}} \left[1 - \frac{\beta}{c_1 n^\gamma} + O(\frac{1}{n^{2\gamma}})\right] \\
= \quad & -\frac{\beta\gamma}{c_1^\beta n^{\gamma\beta+1}} + O(\frac{1}{n^{2\gamma+\gamma\beta}}) \\
= \quad & -\frac{\beta\gamma}{c_1^\beta n^{\gamma\beta+1}} + O(\frac{1}{n^{1+\gamma}}).
\end{aligned} \tag{21}
$$

Thus, thanks to (21) and (16) , there exists $k > 0$ such that for n large enough, we have

$$
\left| \frac{1}{(1+t_{n+1})^\beta} - \frac{1}{(1+t_n)^\beta} \right| \leq \frac{k}{n^{\gamma\beta+1}}.
$$

Using this we get

$$
\begin{aligned}
& \frac{\varepsilon}{(1+t_{n+1})^\beta} < \nabla F(u_{n+1}), v_n > - \frac{\varepsilon}{(1+t_n)^\beta} < \nabla F(u_{n+1}), v_n > \\
\leq \; & \frac{k\varepsilon}{n^{\gamma\beta+1}} \|\nabla F(u_{n+1})\| \|v_n\| \\
\leq \; & \frac{k\varepsilon}{n^{\gamma\beta+1}} \|\nabla F(u_{n+1})\| \|v_{n+1} + \Delta t_n a(t_{n+1}) v_{n+1} + \Delta t_n \nabla F(u_{n+1})\| \text{ by (7)} \\
\leq \; & \varepsilon \left(\frac{k}{n^{\gamma\beta+1}} \|\nabla F(u_{n+1})\| \|v_{n+1}\| + \frac{k \Delta t_n a(t_{n+1})}{n^{\gamma\beta+1}} \|\nabla F(u_{n+1})\| \|v_{n+1}\| + \frac{k\Delta t_n}{n^{\gamma\beta+1}} \|\nabla F(u_{n+1})\|^2 \right) \text{ by (7)}
\end{aligned}
$$

and then

$$
\begin{aligned}
& \Phi_\varepsilon(u_n, v_n) - \Phi_\varepsilon(u_{n+1}, v_{n+1}) \\
\geq \; & \eta(\Delta t_n)^2 \|v_{n+1}\|^2 + \frac{\varepsilon \Delta t_n}{(1+t_{n+1})^\beta} \|\nabla F(u_{n+1})\|^2 - \frac{\varepsilon \Delta t_n a(t_{n+1})}{(1+t_{n+1})^\beta} \|\nabla F(u_{n+1})\| \|v_{n+1}\| - \qquad (22) \\
& - \frac{\varepsilon c_F \Delta t_n}{(1+t_n)^\beta} \|v_{n+1}\|^2 - \frac{\varepsilon c_F (\Delta t_n)^2 a(t_{n+1})}{(1+t_n)^\beta} \|v_{n+1}\|^2 - \frac{\varepsilon c_F (\Delta t_n)^2}{(1+t_n)^\beta} \|v_{n+1}\| \|\nabla F(u_{n+1})\| - \\
& \frac{k\varepsilon}{n^{\gamma\beta+1}} \|\nabla F(u_{n+1})\| \|v_{n+1}\| - \frac{k\varepsilon \Delta t_n a(t_{n+1})}{n^{\gamma\beta+1}} \|\nabla F(u_{n+1})\| \|v_{n+1}\| - \frac{k\varepsilon\Delta t_n}{n^{\gamma\beta+1}} \|\nabla F(u_{n+1})\|^2.
\end{aligned}
$$

By using (17) we get

$$
\begin{aligned}
\frac{\varepsilon \Delta t_n}{(1+t_{n+1})^\beta} &\geq \varepsilon \delta_1 (\Delta t_n)^2, \\
\frac{\varepsilon c_F \Delta t_n}{(1+t_n)^\beta} &\leq \varepsilon c_F \delta_2 (\Delta t_n)^2.
\end{aligned}
$$

Combining (17) and (13), we deduce that

$$
\begin{aligned}
\frac{\varepsilon \Delta t_n a(t_{n+1})}{(1+t_{n+1})^\beta} &\leq \varepsilon \delta_2 (\Delta t_n)^2, \\
\frac{\varepsilon c_F (\Delta t_n)^2 a(t_{n+1})}{(1+t_n)^\beta} &\leq \varepsilon c_F (\Delta t_n)^2, \\
\frac{\varepsilon c_F (\Delta t_n)^2}{(1+t_n)^\beta} &\leq \varepsilon c_F (\Delta t_n)^2.
\end{aligned}
$$

Using (16) and (15), we get for n large enough

$$
\begin{aligned}
\frac{k\varepsilon}{n^{\gamma\beta+1}} &\leq \varepsilon \delta_1 (\Delta t_n)^2, \\
\frac{k\varepsilon \Delta t_n a(t_{n+1})}{n^{\gamma\beta+1}} &\leq \varepsilon \delta_1 (\Delta t_n)^2, \\
\frac{k\varepsilon \Delta t_n}{n^{\gamma\beta+1}} &\leq \varepsilon \frac{\delta_1}{4} (\Delta t_n)^2.
\end{aligned}
$$

Then (22) becomes for n large enough

$$\begin{aligned}
&\Phi_\varepsilon(u_n,v_n)-\Phi_\varepsilon(u_{n+1},v_{n+1})\\
\geq\ &[\eta-\varepsilon c_F\delta_2-\varepsilon c_F](\Delta t_n)^2\|v_{n+1}\|^2+\varepsilon\frac{3\delta_1}{4}(\Delta t_n)^2\|\nabla F(u_{n+1})\|^2\\
&-\varepsilon[\delta_2+c_F+2\delta_1](\Delta t_n)^2\|\nabla F(u_{n+1})\|\|v_{n+1}\|.
\end{aligned}$$

Using Young's inequality, then there is a positive constant $K=K(\delta_1,\delta_2,c_F)$ such that

$$[\delta_2+c_F+2\delta_1]\|\nabla F(u_{n+1})\|\|v_{n+1}\|\leq K\|v_{n+1}\|^2+\frac{\delta_1}{4}\|\nabla F(u_{n+1})\|.$$

So we deduce for n large enough

$$\begin{aligned}
&\Phi_\varepsilon(u_n,v_n)-\Phi_\varepsilon(u_{n+1},v_{n+1})\\
\geq\ &(\Delta t_n)^2[\eta-\varepsilon(c_F\delta_2+c_F+K)]\|v_{n+1}\|^2+\varepsilon\frac{\delta_1}{2}(\Delta t_n)^2\|\nabla F(u_{n+1})\|^2. \qquad (23)
\end{aligned}$$

Now we choose $\varepsilon=\varepsilon_1$ small enough such that

$$\varepsilon(c_F\delta_2+c_F+K)\leq\frac{\eta}{2}.$$

Then (23) becomes

$$\Phi_{\varepsilon_1}(u_n,v_n)-\Phi_{\varepsilon_1}(u_{n+1},v_{n+1})\geq\frac{\eta}{2}(\Delta t_n)^2\|v_{n+1}\|^2+\varepsilon_1\frac{\delta_1}{2}(\Delta t_n)^2\|\nabla F(u_{n+1})\|^2.$$

The proof of lemma 3.4 is completed by taking $\delta=\min\left(\frac{\eta}{2},\varepsilon_1\frac{\delta_1}{2}\right)$. □

End of the proof of theorem 3.2. Thanks to lemma 3.4 and the fact that $\lim_{n\to+\infty}\Phi_{\varepsilon_1}(u_n,v_n)=\lim_{n\to+\infty}E(u_n,v_n)$ exists, the sum $\sum(\Delta t_n)^2[\|v_{n+1}\|+\|\nabla F(u_{n+1})\|]^2$ converges. Now since $\sum(\Delta t_n)^2$ diverges, there exists some injection $\varphi:\mathbb{N}\longrightarrow\mathbb{N}$ such that

$$\lim_{n\to+\infty}\|v_{\varphi(n)}\|+\|\nabla F(u_{\varphi(n)})\|=0.$$

Up to a subsequence, we may assume that $(u_{\varphi(n)})$ converges. Let a_φ be its limit. Clearly we have $(a_\varphi,0)\in\omega((u_n,v_n)_{n\in\mathbb{N}})$ and $\nabla F(a_\varphi)=0$. By (19) we deduce that $a_\varphi\in\operatorname{argmin}F$. Now, using the point 4. of the proposition 3.1, we get that for all $(a,b)\in\omega((u_n,v_n)_{n\in\mathbb{N}})$, we have

$$\frac{1}{2}\|b\|^2+F(a_\varphi)\leq\frac{1}{2}\|b\|^2+F(a)=F(a_\varphi).$$

Then $b=0$ and $a\in\operatorname{argmin}F=\mathcal{S}$. Therefore 0 is the only cluster point of (v_n) and $(\nabla F(u_n))$, therefore

$$\lim_{n\to+\infty}\|v_n\|+\|\nabla F(u_n)\|=0.$$

□

In the sequel we assume that there exists $\theta \in (0, \frac{1}{2}]$ such that

$$\forall a \in \mathcal{S}\ \exists r_a > 0\ \exists \nu_a > 0/\ \forall u \in \mathbb{R}^d:\ \|u-a\| < r_a \Longrightarrow \|\nabla F(u)\| \geq \nu_a |F(u)-F(a)|^{1-\theta}. \quad (24)$$

Remark 3.5. ([17, 18, 10, 8]) Assumption (24) is satisfied if one of the following two cases holds:
- F is a polynomial, or
- F is analytic and $\mathcal{S}$ is compact.

For the sequel, we need the following well-known result.

Lemma 3.6. *([3, 11]) Let Γ be a compact and connected subset of $\mathcal{S}$. Then we have*

(1) F assumes a constant value on Γ. We denote by $\bar{F}$ the common value of $F(a)$, $a \in \Gamma$.

(2) There exist $r > 0$ and $\nu > 0$ such that

$$dist(u, \Gamma) < r \Longrightarrow \|\nabla F(u)\| \geq \nu |F(u) - \bar{F}|^{1-\theta}.$$

The main result of this paper is the following:

Theorem 3.7. *We assume that the hypotheses of the theorem 3.2 are true and that F also satisfies (24) and $\beta \in (0, \frac{\theta}{1-\theta})$. Let (u_n, v_n) be a sequence satisfying (7) such that (u_n) is bounded. Then there exists $a \in \mathcal{S}$ such that*

$$\lim_{n \to +\infty} \|v_n\| + \|u_n - a\| = 0.$$

In addition as $n \to +\infty$ we have

$$\|u_n - a\| = O\left(t_n^{-\frac{\theta-\beta(1-\theta)}{1-2\theta}}\right) = O\left(n^{-\frac{\theta-\beta(1-\theta)}{(1-2\theta)(1+\beta)}}\right) \quad (25)$$

Proof. From proposition 3.1, we know that $\omega((u_n, v_n)_{n\in\mathbb{N}})$ is a non-empty compact, connected set. We also know that $\lim_{n\to+\infty} \|v_n\| = 0$ and that $\omega((u_n, v_n)_{n\in\mathbb{N}}) \subset \times\{0\}$ (see theorem 3.2). Let $\Gamma = \{a/\ (a, 0) \in \omega((u_n, v_n)_{n\in\mathbb{N}})\}$ and $K = \lim_{n\to+\infty} E(u_n, v_n)$.
Let's put $H_n = \Phi_{\varepsilon_1}(u_n, v_n) - K$ where Φ_{ε_1} as in lemma 3.4. Then $(H_n)_{n \geq N_1}$ in non-increasing, tends to 0 (and then positive). Assume that for some $n_0 \geq N_1$, $H_{n_0} = K$, then for all $n \geq n_0$, $H_n = K$. According to (20), $v_{n+1} = 0$ for all $n \geq n_0$ and then $u_n = u_{n_0}$ for all $n \geq n_0$.

Otherwise, since we may assume that $K = 0$ (so that we also have that $\forall a \in \Gamma$, $F(a) = 0$), there exists some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, one has $0 < H_n \leq 1$.
It is easy to see that

$$\lim_{n\to+\infty} \mathrm{dist}((u_n), \Gamma) = 0.$$

Applying lemma 3.6, then there exist $r > 0$ and $\nu > 0$ such that

$$\forall u \in \mathbb{R}^d \quad dist(u, \Gamma) < r \Longrightarrow \|\nabla F(u)\| \geq \nu |F(u)|^{1-\theta}.$$

Since $\Gamma = \{a/\ (a,0) \in \omega((u_n, v_n)_{n\in\mathbb{N}})\}$, then there exists $N \geq N_1$ such that for all $n \geq N$, $\mathrm{dist}(u_n, \Gamma) \leq r$. Then for all $n \geq N$

$$\|\nabla F(u_n)\| \geq \nu |F(u_n)|^{1-\theta}. \tag{26}$$

As in [21, 2], let $n \geq N$ such that

$$H_{n+1} > \frac{H_n}{2}. \tag{27}$$

We have

$$\begin{aligned} H_n^\theta - H_{n+1}^\theta &= \int_{H_{n+1}}^{H_n} \theta s^{\theta-1}\, ds \\ &\geq \theta[H_n - H_{n+1}] H_n^{\theta-1} \\ &\underset{\text{by (27)}}{\geq} \theta 2^{\theta-1}[H_n - H_{n+1}] H_{n+1}^{\theta-1}. \end{aligned} \tag{28}$$

For all $n \geq N$, we have

$$\begin{aligned} H_{n+1}^{1-\theta} &= \left| \frac{1}{2}\|v_{n+1}\|^2 + F(u_{n+1}) + \frac{\varepsilon_1}{(1+t_{n+1})^\beta} < \nabla F(u_{n+1}), v_{n+1} > \right|^{1-\theta} \\ &\leq \frac{1}{2^{1-\theta}}\|v_{n+1}\|^{2(1-\theta)} + |F(u_{n+1})|^{1-\theta} + \varepsilon_1^{1-\theta}\|\nabla F(u_{n+1})\|^{1-\theta}\|v_{n+1}\|^{1-\theta}. \end{aligned} \tag{29}$$

Thanks to Young's inequality we obtain

$$\|\nabla F(u_{n+1})\|^{1-\theta}\|v_{n+1}\|^{1-\theta} \leq \|\nabla F(u_{n+1})\| + \|v_{n+1}\|^{\frac{1-\theta}{\theta}}$$

According to (26), we get for all $n \geq N$

$$|F(u_{n+1})|^{1-\theta} \leq \frac{1}{\nu}\|\nabla F(u_{n+1})\|.$$

Then (29) becomes (using the fact that $v_n \to 0$, $\frac{1-\theta}{\theta} \geq 1$ and $2(1-\theta) \geq 1$)

$$\begin{aligned} H_{n+1}^{1-\theta} &\leq \left(\frac{1}{2^{1-\theta}} + \varepsilon_1^{1-\theta}\right)\|v_{n+1}\| + \left(\frac{1}{\nu} + \varepsilon_1^{1-\theta}\right)\|\nabla F(u_{n+1})\| \\ &\leq C\left(\|v_{n+1}\| + \|\nabla F(u_{n+1})\|\right) \end{aligned} \tag{30}$$

where $C = \max\left(\frac{1}{2^{1-\theta}} + \varepsilon_1^{1-\theta}, \frac{1}{\nu} + \varepsilon_1^{1-\theta}\right)$. By using (30) and (20), (28) becomes

$$H_n^\theta - H_{n+1}^\theta \geq \frac{\delta}{C}(\Delta t_n)^2\left(\|v_{n+1}\| + \|\nabla F(u_{n+1})\|\right).$$

Then we get

$$(\Delta t_n)^2\left(\|v_{n+1}\| + \|\nabla F(u_{n+1})\|\right) \leq \frac{C}{\delta}[H_n^\theta - H_{n+1}^\theta]. \tag{31}$$

On the other hand, let $n \geq N$ such that

$$H_{n+1} \leq \frac{H_n}{2}. \tag{32}$$

From (20), we get
$$\delta(\Delta t_n)^2[\|v_{n+1}\| + \|\nabla F(u_{n+1})\|]^2 \leq H_n - H_{n+1}.$$

Then we deduce

$$\begin{aligned}
\Delta t_n\|v_{n+1}\| &\leq \frac{1}{\delta^{\frac{1}{2}}}[H_n - H_{n+1}]^{\frac{1}{2}} \leq \frac{1}{\delta^{\frac{1}{2}}}[H_n]^{\frac{1}{2}} \\
&\leq \frac{1}{\delta^{\frac{1}{2}}}[H_n]^{\theta} \qquad (H_n \leq 1) \\
&\underset{\text{by (32)}}{\leq} \frac{1}{\delta^{\frac{1}{2}}}\frac{2^\theta}{2^\theta - 1}\left(H_n^\theta - H_{n+1}^\theta\right).
\end{aligned} \tag{33}$$

Since $\Delta t_n \sim \frac{\gamma c_1}{n^{1-\gamma}}$, then we can assume that for all $n \geq N$

$$\frac{1}{2(1+\beta)}\frac{c_1}{n^{\frac{\beta}{1+\beta}}} = \frac{1}{2}\frac{\gamma c_1}{n^{1-\gamma}} \leq \Delta t_n \leq \frac{2\gamma c_1}{n^{1-\gamma}} = \frac{2}{1+\beta}\frac{c_1}{n^{\frac{\beta}{1+\beta}}} \leq 1. \tag{34}$$

where we used that $\gamma = \frac{1}{1+\beta}$.
Then from (33) we deduce

$$(\Delta t_n)^2\|v_{n+1}\| \leq \frac{1}{\delta^{\frac{1}{2}}}\frac{2^\theta}{2^\theta - 1}\left(H_n^\theta - H_{n+1}^\theta\right).$$

In both cases, we have for all $n \geq N$

$$(\Delta t_n)^2\|v_{n+1}\| \leq C_1\left(H_n^\theta - H_{n+1}^\theta\right) \tag{35}$$

where $C_1 = \max\left(\frac{C}{\delta}, \frac{1}{\delta^{\frac{1}{2}}}\frac{2^\theta}{2^\theta-1}\right)$.

Now we define the map $G : s \longmapsto \frac{1}{1-2\theta}s^{2\theta-1}$.
For $n \geq N$ such that $H_{n+1} > \frac{H_n}{2}$. We have

$$\begin{aligned}
G(H_{n+1}) - G(H_n) &= \int_{H_{n+1}}^{H_n} s^{2\theta-2}ds \\
&\geq (H_n - H_{n+1})H_n^{-2+2\theta} \\
&\geq 2^{-2+2\theta}(H_n - H_{n+1})H_{n+1}^{-2+2\theta} \\
&\underset{\text{by (30)}}{\geq} \frac{2^{-2+2\theta}}{C^2}\frac{H_n - H_{n+1}}{(\|v_{n+1}\| + \|\nabla F(u_{n+1})\|)^2} \\
&\underset{\text{by (20)}}{\geq} \frac{2^{-2+2\theta}}{C^2}\delta(\Delta t_n)^2.
\end{aligned}$$

On the other hand, for $n \geq N$ such that $H_{n+1} \leq \frac{H_n}{2}$, we have

$$\begin{aligned}
G(H_{n+1}) - G(H_n) &= \frac{1}{1-2\theta}\left(H_{n+1}^{2\theta-1} - H_n^{2\theta-1}\right) \\
&\geq \frac{1}{1-2\theta}\left(\frac{H_n^{2\theta-1}}{2^{2\theta-1}} - H_n^{2\theta-1}\right) \\
&\geq \frac{2^{1-2\theta}-1}{1-2\theta} H_n^{2\theta-1} \\
&\geq \frac{2^{1-2\theta}-1}{1-2\theta} \quad (H_n \leq 1) \\
&\geq \frac{2^{1-2\theta}-1}{1-2\theta}(\Delta t_n)^2. \quad (\Delta t_n \leq 1).
\end{aligned}$$

Then for all $n \geq N$, we have

$$G(H_{n+1}) - G(H_n) \geq C_2 (\Delta t_n)^2 \tag{36}$$

where $C_2 = \min\left(\frac{2^{-2+2\theta}\delta}{C^2}, \frac{2^{1-2\theta}-1}{1-2\theta}\right) > 0$.
Let $n \geq N+1$. Summing (36) from N to $n-1$, and by using (34), it comes

$$\begin{aligned}
G(H_n) - G(H_N) &\geq \sum_{k=N}^{n-1} C_2 (\Delta t_k)^2 \\
&\geq C_2 \frac{c_1}{2(1+\beta)} \sum_{k=N}^{n-1} \frac{1}{k^{\frac{2\beta}{1+\beta}}} \\
&\geq C_2 \frac{c_1}{2(1+\beta)} \int_N^n \frac{ds}{s^{\frac{2\beta}{1+\beta}}} \\
&\geq C_2 \frac{c_1(1-\beta)}{2(1+\beta)^2}\left[n^{\frac{1-\beta}{1+\beta}} - N^{\frac{1-\beta}{1+\beta}}\right].
\end{aligned}$$

Let's pose $C_3 = C_2 \frac{c_1(1-\beta)}{2(1+\beta)^2}$. We have successively

$$\begin{aligned}
& G(H_n) = \frac{1}{1-2\theta} {H_n}^{2\theta-1} \geq G(H_N) + C_3\left(n^{\frac{1-\beta}{1+\beta}} - N^{\frac{1-\beta}{1+\beta}}\right), \\
& H_n^{2\theta-1} \geq (1-2\theta)\left[G(H_N) + C_3\left(n^{\frac{1-\beta}{1+\beta}} - N^{\frac{1-\beta}{1+\beta}}\right)\right], \\
& H_n^{2\theta-1} \geq \left[(1-2\theta)G(H_N) - (1-2\theta)C_3 N^{\frac{1-\beta}{1+\beta}} + (1-2\theta)C_3 n^{\frac{1-\beta}{1+\beta}}\right].
\end{aligned}$$

Thus for all $n \geq N+1$, we get

$$H_n \leq \frac{1}{\left[(1-2\theta)G(H_N) - (1-2\theta)C_3 N^{\frac{1-\beta}{1+\beta}} + (1-2\theta)C_3 n^{\frac{1-\beta}{1+\beta}}\right]^{\frac{1}{1-2\theta}}}.$$

Then there exists a constant $\lambda > 0$ such that for all $n \geq N+1$

$$H_n \leq \frac{\lambda}{n^{\frac{1-\beta}{(1+\beta)(1-2\theta)}}}. \tag{37}$$

Let $n \in \mathbb{N}$ such that $2^n \geq N$. By using (20), we get

$$H_{2^n} - H_{2^{n+1}} = \sum_{k=2^n}^{2^{n+1}-1} H_k - H_{k+1} \geq \delta \sum_{k=2^n}^{2^{n+1}-1} (\Delta t_k)^2 \|v_{k+1}\|^2. \tag{38}$$

Since for n large enough (Δt_n) is non-increasing, we obtain

$$\begin{aligned}\sum_{k=2^n}^{2^{n+1}-1} (\Delta t_k)^2 \|v_{k+1}\|^2 &\geq (\Delta t_{2^{n+1}-1}) \sum_{k=2^n}^{2^{n+1}-1} (\Delta t_k) \|v_{k+1}\|^2 \\ &\geq \frac{c_1}{2(1+\beta)} \frac{1}{(2^{n+1}-1)^{\frac{\beta}{1+\beta}}} \sum_{k=2^n}^{2^{n+1}-1} (\Delta t_k) \|v_{k+1}\|^2. \end{aligned} \tag{39}$$

Combining (39) and (38) we get

$$\begin{aligned}\sum_{k=2^n}^{2^{n+1}-1} (\Delta t_k) \|v_{k+1}\|^2 &\leq \frac{2(1+\beta)}{c_1} (2^{n+1}-1)^{\frac{\beta}{1+\beta}} \frac{1}{\delta} [H_{2^n} - H_{2^{n+1}}] \\ &\leq C_4 2^{\frac{n\beta}{1+\beta}} H_{2^n} \end{aligned} \tag{40}$$

where $C_4 = \frac{2(1+\beta)}{c_1 \delta} (2^{\frac{\beta}{1+\beta}} + 1)$.
On the other hand, using the Cauchy-Schwarz inequality, we get

$$\sum_{k=2^n}^{2^{n+1}-1} (\Delta t_k) \|v_{k+1}\| \leq \left(\sum_{k=2^n}^{2^{n+1}-1} \Delta t_k \right)^{\frac{1}{2}} \left(\sum_{k=2^n}^{2^{n+1}-1} (\Delta t_k) \|v_{k+1}\|^2 \right)^{\frac{1}{2}}. \tag{41}$$

$$\begin{aligned}\sum_{k=2^n}^{2^{n+1}-1} \Delta t_k &\leq \sum_{k=2^n}^{2^{n+1}-1} \frac{2}{1+\beta} \frac{c_1}{k^{\frac{\beta}{1+\beta}}} \\ &\leq C_4 \int_{2^n-1}^{2^{n+1}-1} \frac{dt}{t^{\frac{\beta}{1+\beta}}} \end{aligned}$$

where $C_4 = \frac{2c_1}{1+\beta}$. Now, a straightforward calculation gives

$$\int_{2^n-1}^{2^{n+1}-1} \frac{dt}{t^{\frac{\beta}{1+\beta}}} = (1+\beta) \left[(2^{n+1}-1)^{\frac{1}{1+\beta}} - (2^n - 1)^{\frac{1}{1+\beta}} \right] \leq (1+\beta) 2^{n \frac{1}{1+\beta}} \left[2^{\frac{1}{1+\beta}} + 1 \right]$$

$$\sum_{k=2^n}^{2^{n+1}-1} \Delta t_k \leq C_4(1+\beta)2^{n\frac{1}{1+\beta}}\left[2^{\frac{1}{1+\beta}}+1\right]$$

and then

$$\left(\sum_{k=2^n}^{2^{n+1}-1} \Delta t_k\right)^{\frac{1}{2}} \leq C_5 2^{n\frac{1}{2(1+\beta)}} \tag{42}$$

where $C_5 = \left(C_4(1+\beta)\left[2^{\frac{1}{1+\beta}}+1\right]\right)^{\frac{1}{2}}$.
Using (41), (42), (40) and (37) leads to

$$\begin{aligned}\sum_{k=2^n}^{2^{n+1}-1} (\Delta t_k)\|v_{k+1}\| &\leq C_5 2^{n\frac{1}{2(1+\beta)}}\left(C_4 2^{\frac{n\beta}{1+\beta}} H_{2^n}\right)^{\frac{1}{2}} \\ &\leq C_5\sqrt{C_4\lambda}(2^n)^{\frac{1}{2(1+\beta)}+\frac{\beta}{2+2\beta}-\frac{1-\beta}{2(1+\beta)(1-2\theta)}} \\ &\leq C_5\sqrt{C_4\lambda}(2^n)^{-\frac{\theta-\beta(1-\theta)}{(1+\beta)(1-2\theta)}}.\end{aligned}$$

Yet by assumption $\beta \in (0, \frac{\theta}{1-\theta})$, thus $\sum(\Delta t_k)\|V_{k+1}\|$ converges. □

4 Numerical simulations

As in [15], we present some numerical simulations with specific F that do not necessarily satisfy our results' assumptions.

The implicit sequence satisfying (7) is constructed via a quasi-newton algorithm.

4.1 A convex and coercive function F

The function F that we consider is

$$F(x,y) = ((x-1)^2 + (y-1)^2 - 1)^2 \chi_{(x-1)^2+(y-1)^2-1\geq 0},$$

and we take

$$a(t) = \frac{1}{(1+t)^\beta},\ \beta = 0.1.$$

In figure (1), we observe the numerical decrease of E.
In figure (2), we observe the numerical convergence of the sequence satisfying (7).

4.2 A non-convex and coercive function F

The function F that we consider is

$$F(x,y) = (x^2+y^2-1)^2,$$

Figure 1: Simulation for (7).

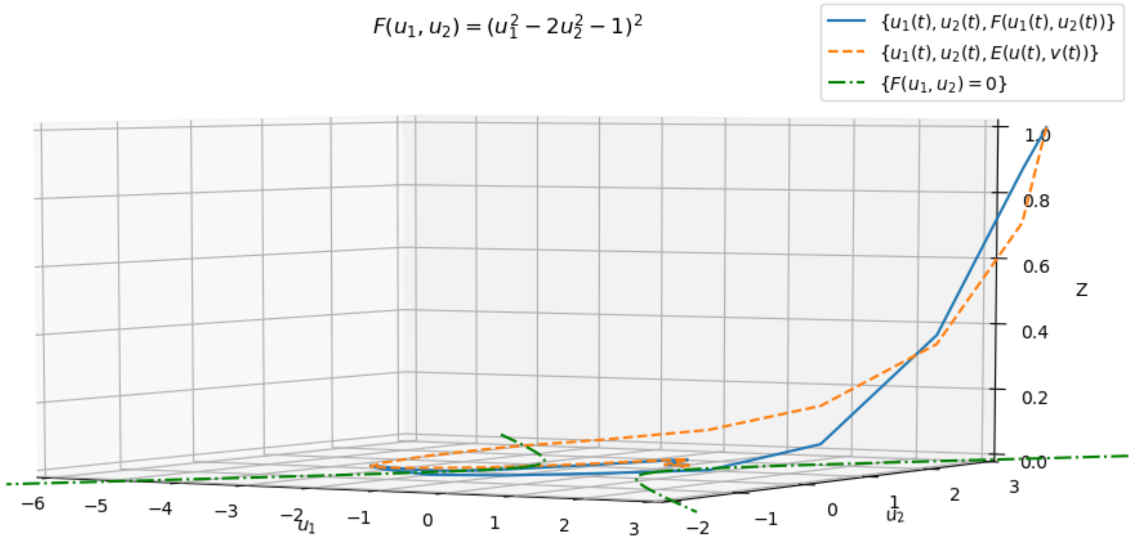
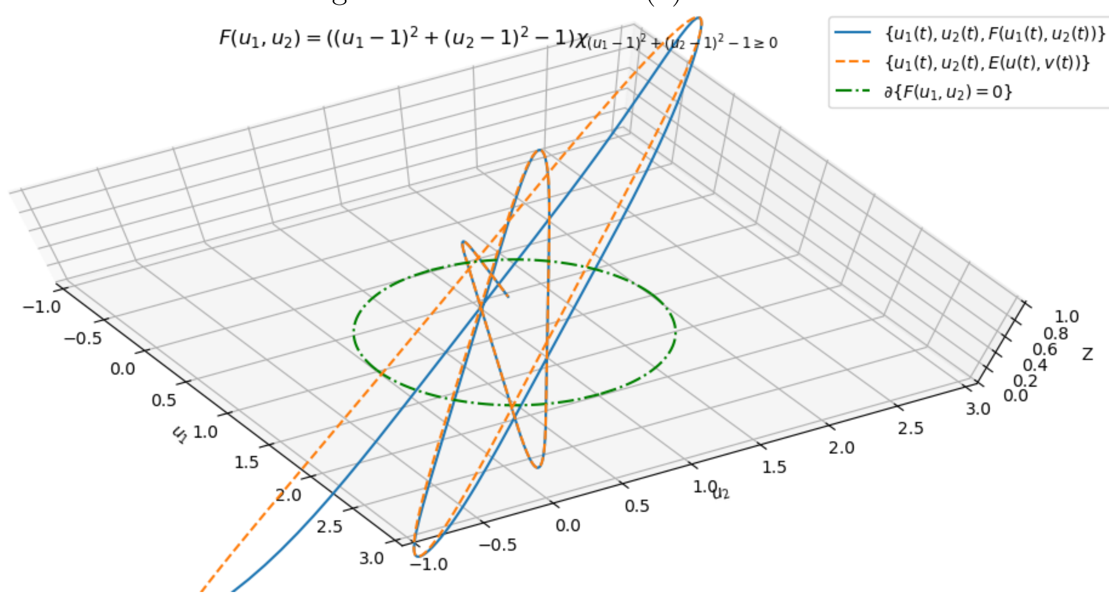


Figure 2: Simulation for (7).

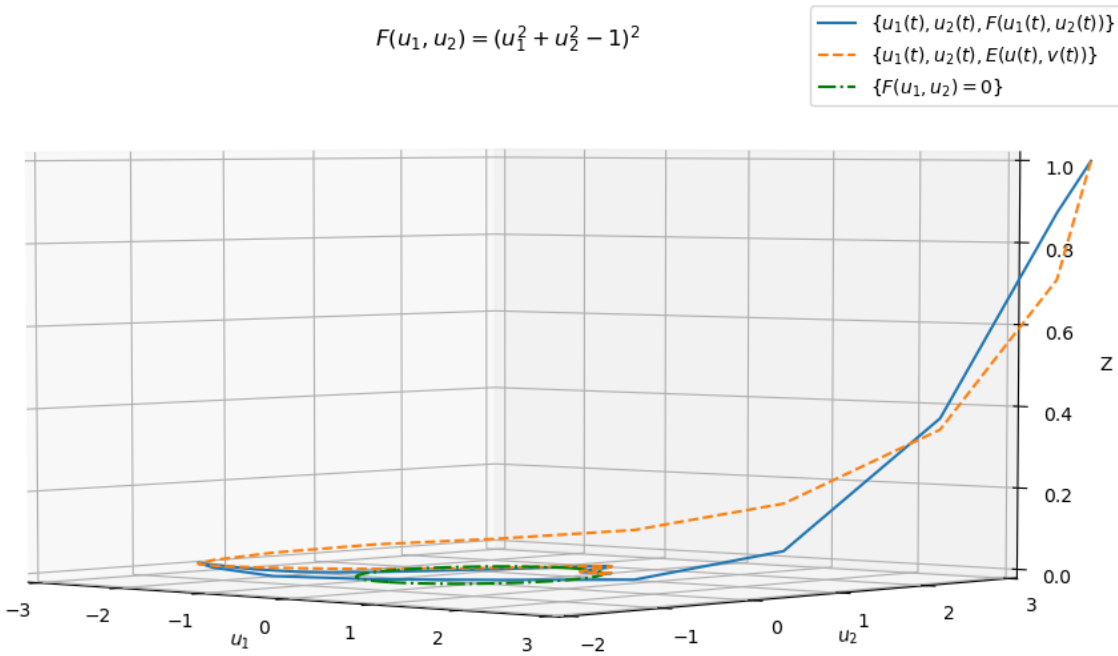


and we take again

$$a(t) = \frac{1}{(1+t)^{\beta}}, \ \beta = 0.1.$$

In figure (3), we observe also the numerical decrease of E.
In figure (4), we observe the numerical convergence of the sequence satisfying (7).

Figure 3: Simulation for (7).



4.3 A non-convex and non-coercive function F

The function F that we consider here is

$$F(x, y) = (x^2 - 2y^2 - 1)^2,$$

and we take again

$$a(t) = \frac{1}{(1+t)^\beta},\ \beta = 0.1.$$

In figure (5), we also observe the numerical decrease of E.
In figure (6), we also observe the numerical convergence of the sequence satisfying (7).

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Figure 4: Simulation for (7).

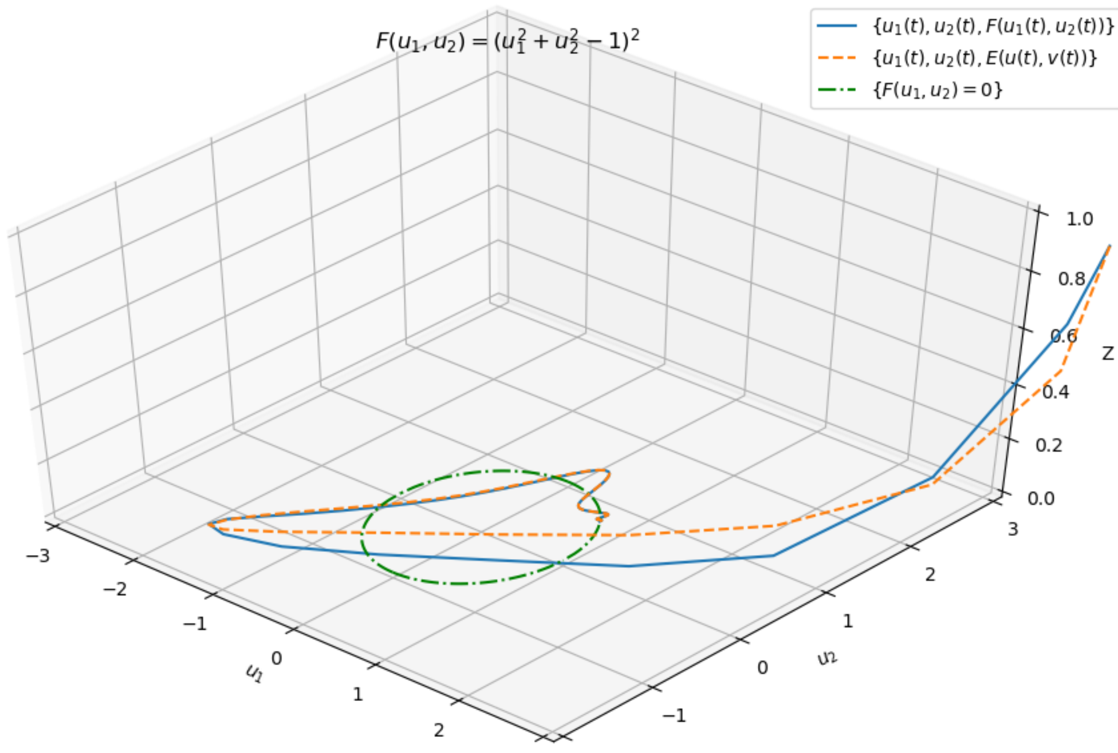

Figure 5: Simulation for (7).

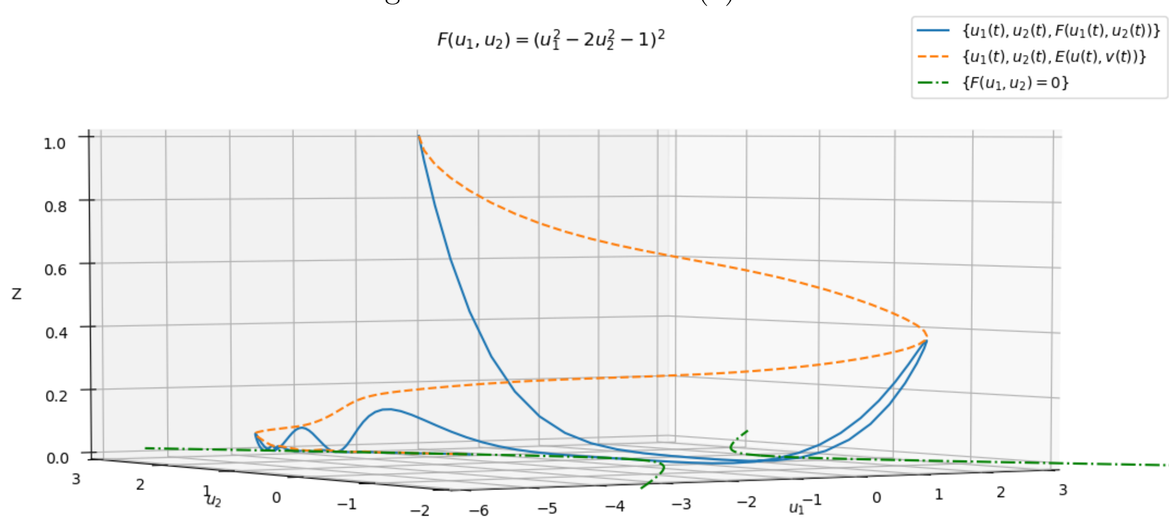

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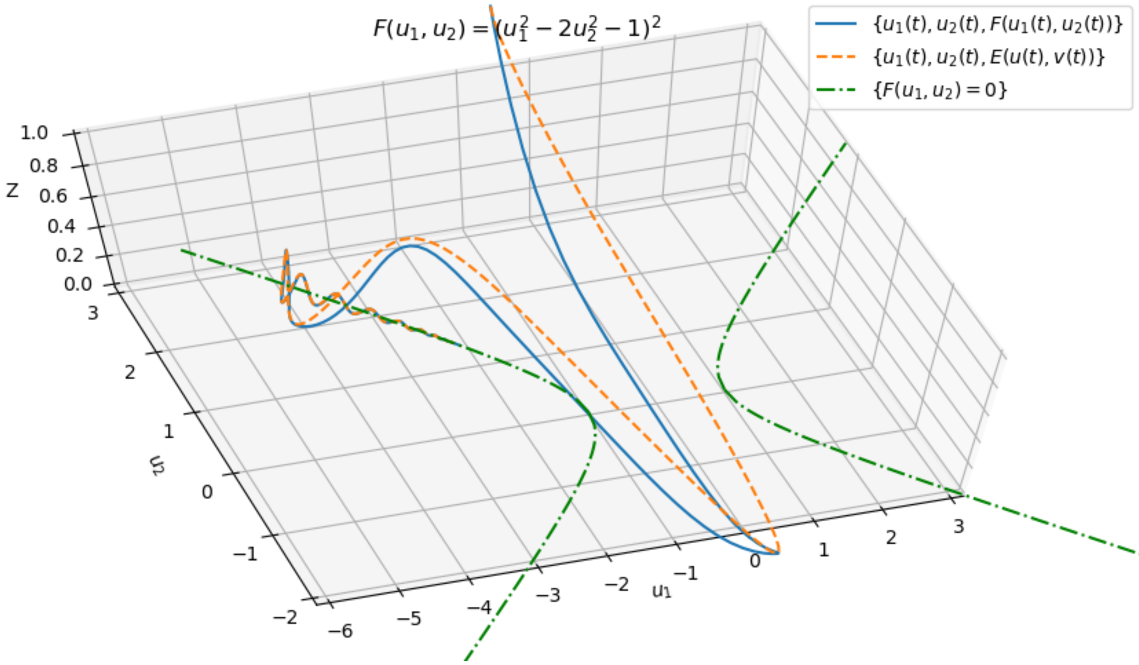
Figure 6: Simulation for (7).



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